Introductory Multivariable Analytic Combinatorics

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A Course on Analytic Combinatorics

Objectives

Develop a combinatorial understanding of various function classes, esp. algebraic, and D-finite

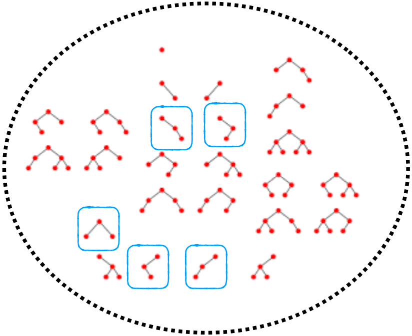
Examine singularities of multivariable combinatorial generating functions and understand the relationship between geometry and the coefficient asymptotics.

Organization

- I. Combinatorial Functional Equations and Taxonomy
- II. Singularities and Critical Points
- III. Diagonal Asymptotics

I. Combinatorial Functional Equations

Combinatorial Classes



tree $\mapsto z^{\#\text{nodes}}$

$C(z) = z + 2z^2 + 5z^3 + 14z^4 + 42z^5$

Given a class $\mathcal{C}$, and size $|\cdot|$

$$C(z) := \sum_{\gamma \in \mathcal{C}} z^{|\gamma|}$$

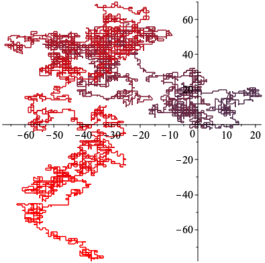
$$C(z) = \sum_{n \geq 0} c_n z^n$$

$c_n = \#$ objects of size n

$C(z)$ is the ordinary generating function (OGF) for $\mathcal{C}$

Objectives

- Enumerate objects exactly or asymptotically $c_n = ?$
- Understand the large scale behaviour of the objects in a class
- Interpret functional equations combinatorially
- Answer the question:
  Under which conditions does the OGF of a combinatorial class satisfy a linear ODE with polynomial coefficients?
  D-finite/ Holonomic/ G-functions



Everything is non-holonomic unless it is holonomic by design.

Flajolet, Gerhold and Salvy

Combinatorial Calculus

	$\mathcal{C}$	Notes	$C(z) = \sum z^{\lvert\gamma\rvert}$
Epsilon	$\{\epsilon\}$	$\lvert\epsilon\rvert = 0$	1
Atom	$\{\circ\}$	$\lvert\circ\rvert = 1$	z
Disjoint Union	$\mathcal{A} + \mathcal{B}$	$\gamma \times \epsilon_{\mathcal{A}}, \gamma \times \epsilon_{\mathcal{B}}$	$A(z) + B(z)$
Cartesian Product	$\mathcal{A} \times \mathcal{B}$	$(\alpha, \beta), \alpha \in \mathcal{A}, \beta \in \mathcal{B}$	$A(z)B(z)$
Power	$\mathcal{A}^k$	$(\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathcal{A}$	$A(z)^k$
Sequence	$\mathrm{Seq}(\mathcal{A}) = \mathcal{A}^*$	$\epsilon + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \ldots$	$\frac{1}{1-A(z)}$

Binary Words $\{\epsilon, \circ, \bullet, \circ\circ, \circ\bullet, \bullet\circ, \bullet\bullet, \circ\circ\circ, \ldots\}$

$$
\begin{array}{ccccccccc}
\{\circ\} & \{\bullet\} & \mathcal{A} & = & \{\circ, \bullet\} & \mathcal{C} & = & \mathcal{A}^* \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
z & z & A(z) & = & 2z & C(z) & = & \frac{1}{1-A(z)}
\end{array}
$$

$$\Longrightarrow \quad C(z) = \frac{1}{1-2z}$$

Combinatorial Calculus

	$\mathcal{C}$	Notes	$C(z) = \sum z^{\lvert\gamma\rvert}$
Epsilon	$\{\epsilon\}$	$\lvert\epsilon\rvert = 0$	1
Atom	$\{\circ\}$	$\lvert\circ\rvert = 1$	z
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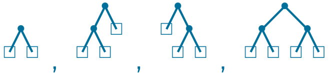
Binary Trees $\{\square,$  $, \ldots\}$

$$
\begin{array}{ccccccccc}
\{\bullet\} & \{\square\} & \mathcal{B} & = & \square & + & \bullet & \times & \mathcal{B}^2 \\
\downarrow & \downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & z & B(z) & = & z & + & 1 & \cdot & B(z)^2
\end{array}
$$

$$\Longrightarrow \quad B(z) = \frac{1-\sqrt{1-4z}}{2z}$$

Specifications

Generically we specify a combinatorial class by a set of combinatorial equations (like we have just seen):

$$\begin{aligned}\mathcal{C}_1 &= \Phi_1(\mathcal{Z}, \mathcal{C}_1, \ldots, \mathcal{C}_r)\\ &\vdots\\ \mathcal{C}_r &= \Phi_r(\mathcal{Z}, \mathcal{C}_1, \ldots, \mathcal{C}_r).\end{aligned} \tag{1}$$

.... and deduce a system of functional equations satisfied by the generating functions:

$$\begin{aligned}C_1(z) &= \Phi_1(z, C_1(z), \ldots, C_r(z))\\ &\vdots\\ C_r(z) &= \Phi_r(z, C_1(z), \ldots, C_r(z)).\end{aligned} \tag{2}$$

Cyclic dependencies change the nature of the generating function.

Acyclic Dependencies: S-regular classes

Combinatorial classes specified using $+, \times, *$, Atoms, and Epsilons with **no** cyclic dependencies are **S-regular classes.**

$$
\begin{aligned}
\mathcal{L} &= \mathrm{Seq}(\{0\} + (\{1\} \times \mathrm{Seq}(\{0\} \times \mathrm{Seq}(\{1\}) \times \{0\}) \times \{1\})) \\
&= (0 + (1(01^*0)^*1))^* = \{\epsilon, 0, 00, 11, 000, 011, 1001, 10101, \dots\}
\end{aligned}
$$

$$
L(z) = \frac{1}{1 - \left(z + z\frac{1}{1 - z\frac{1}{1-z}z}\right)}
$$

Theorem

The generating function of an S-regular class is a **rational** *function.*

Remark: Not all rational functions Taylor series in $\mathbb{N}[[z]]$ arise this way. ($\exists$ singularity criteria)

Cyclic Dependencies: Algebraic Classes

Well defined combinatorial classes specified using $+$, $\times$, Atoms, and Epsilons (using possibly cyclic dependencies) are **algebraic classes.**

$$\mathcal{B} = \square + \bullet \times \mathcal{B} \times \mathcal{B}$$

Theorem

The generating function of an algebraic class is an **algebraic** *function.*

Remark: If a class has a transcendent OGF, it is not an algebraic class.
Remark: Not all algebraic functions with series in $\mathbb{N}[[z]]$ arise this way. ($\exists$ asymptotic criteria)

Derivation Tree

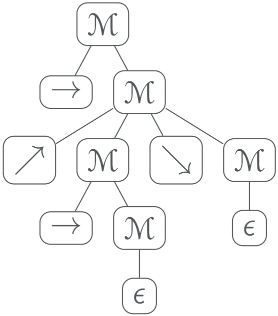
The history of rules expanded is encoded in derivation tree.
We identify derivation trees and elements

Motzkin Paths $\mathcal{M}$

Walks with steps $\{\nearrow, \searrow, \rightarrow\}$ confined to the upper half plane.

$$\mathcal{M} \equiv \epsilon + \rightarrow \mathcal{M} + \nearrow \mathcal{M} \searrow \mathcal{M}.$$

$$\rightarrow \nearrow \rightarrow \searrow \in \mathcal{M}$$



Combinatorial Parameters

A **parameter** of a class is a map $\chi : \mathcal{C} \to \mathbb{Z}$
e.g. $\# \to$ steps ; # leaves in a tree ; end height of a walk

$$C(u,z) := \sum_{\gamma \in \mathcal{C}} u^{\chi(\gamma)} x^{|\gamma|} = \sum_{n \geq 0} \left(\sum_{k \in \mathbb{Z}} c_{k,n} u^k \right) z^n.$$

$c_{k,n} = \#$ objects of size n with parameter value k.
$C(u,z) \in \mathbb{N}[u, u^{-1}][[z]]$ Power series with Laurent polynomial coefficients

Example

$\chi(w) = |w|_\circ = \# \circ$s a word in $\{\circ, \bullet\}^*$: $\chi(\circ \bullet \bullet \circ \bullet) = 2$

$$B(u,z) = 1 + (u+1)\, z + \left(u^2 + 2\, u + 1\right) z^2 + \left(u^3 + 3u^2 + 3u + 1\right) z^3 + \ldots.$$

$$B(u,z) = \left(\frac{1}{1 - (z + uz)} \right). \tag{3}$$

Inherited parameters

The parameter χ is inherited from ξ and ζ if, and only if ...

$\mathcal{C} = \mathcal{A} + \mathcal{B}$

$$\chi(\gamma) = \begin{cases} \xi(\gamma) & \gamma \in \mathcal{A} \\ \zeta(\gamma) & \gamma \in \mathcal{B} \end{cases}$$

$$\implies C(\mathbf{z}) = A(\mathbf{z}) + B(\mathbf{z})$$

$\mathcal{C} = \mathcal{A} \times \mathcal{B}$

$$\chi(\alpha, \beta) = \xi(\alpha) + \zeta(\beta).$$

$$\implies C(\mathbf{z}) = A(\mathbf{z})B(\mathbf{z})$$

e.g. $B(u, z) = \left(\frac{1}{1-(z+uz)}\right)$.

Straightforward translation of structural parameters to OGF

Derived Classes

Given a class $\mathcal{C}$, multidimensional inherited parameter $\chi : \mathcal{C} \to \mathbb{Z}^d$, and vector r, define a derived class of $\mathcal{C}$ as any class

$$\mathcal{C}^{\chi,r} = \cup_n \{\gamma \in \mathcal{C} \mid \chi(\gamma) = (r_1 n, \ldots, r_d n), |\gamma| = n\}$$

Fixed Value

If $r = (0, 0, \ldots, 0)$, then $\chi = (0, 0, \ldots, 0)$: This is the constant term with respect to the non-size variables.

Balanced Subclasses

χ counts occurrences of subobjects; $r = (1, 1, \ldots, 1)$

After two examples, we consider to how find the generating functions of derived classes.

Balanced word classes

$$\mathcal{L} = \{\text{binary expansions of } n \mid n \equiv 0 \mod 3.\}$$ Size = length of string

$$\mathcal{L} = \{\epsilon, \overset{0}{0}, 00, 000, \ldots, \overset{3}{11}, 011, 0011, \ldots, \overset{6}{110}, 0110, 00110, \ldots,$$
$$\overset{9}{1001}, 01001, \overset{12}{1100}, 01100, \ldots, \overset{15}{1111}, 01111, \ldots\}$$

S-regular specification: $\mathcal{L} = (0 + (1(01^*0)^*1))^*$

Parameter: $\chi(w) = (|w|_0, |w|_1) = (\#0s \text{ in } w, \#1s \text{ in } w)$

Balanced sub-class:

$$\begin{aligned}\mathcal{L}_= &= \{w \in \mathcal{L} \mid \chi(w) = (n, n), n \geq 0\} \\ &= \{w \in \mathcal{L} \mid \#0s = \#1s\} \\ &= \{1001, 0011, 0110, 1100, 010101, 101010, 11100001, 10011001, \\ &\qquad 10000111, 00101101, 01011010, 00111001, 00100111, \ldots\}\end{aligned}$$

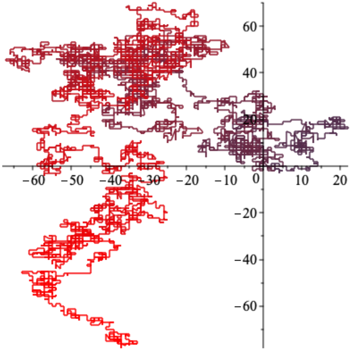
more interesting: $\mathcal{L} \subseteq \{a_1, a_2, \ldots, a_d\}^*$ with $\chi_i(w) = \#$ of a_i in w.

Excursions

$S = \{\uparrow, \downarrow, \leftarrow, \rightarrow\} = ✥$ is a set of steps. Consider walks starting at $(0,0)$ steps from S. Unrestricted walks are S-regular:

$$\{\uparrow, \downarrow, \leftarrow, \rightarrow\}^*$$

Define parameter $\chi(w) :=$ endpoint of w.



Endpoint is an inherited parameter

$$\sum walk^{✥}_{\mathbb{Z}^2}((0,0) \xrightarrow{n} (k,\ell))\, x^k y^\ell t^n = \frac{1}{1 - t(x + 1/x + y + 1/y)}$$

Excursions are a derived class

$$\mathcal{E} = \{w \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}^* \mid \chi(w) = (0,0)\}$$

Diagonals

The central diagonal maps series expansions to series expansions. e.g.

$$\Delta : K[[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]][[t]] \to K[[t]].$$

defined as:

$$\Delta F(\mathbf{z}, t) = \Delta \sum_{k \geq 0} \sum_{\mathrm{n} \in \mathbb{Z}^d} f(\mathbf{n}, k)\, \mathbf{z}^{\mathrm{n}} t^k := \sum_{n \geq 0} f(n, n, \ldots, n)\, t^n. \tag{4}$$

$$\Delta(z_1^2 z_2 t + 3\mathbf{z}_1\mathbf{z}_2\mathbf{t} + 7z_1z_2t^2 + 5\mathbf{z}_1^2\mathbf{z}_2^2\mathbf{t}^2) = 3t + 5t^2$$

Defined for any series.

We use diagonals to describe the generating functions of derived classes.

Example: Multinomials

Central diagonal

$$\Delta \frac{1}{1-x-y} = \Delta \sum_{n\geq 0}(x+y)^n = \Delta \sum_{\ell\geq 0}\sum_{k\geq 0}\binom{\ell+k}{k}x^k y^\ell = \sum_{n\geq 0}\binom{2n}{n,n}y^n.$$

Off center diagonals

$$\Delta^{(r,s)}\frac{1}{1-x-y} = \sum_{n\geq 0}\binom{rn+sn}{rn,sn}y^n.$$

This example generalizes naturally to arbitrary dimension, using multinomials:

$$\Delta^{\mathrm{r}}\frac{1}{1-(z_1+\cdots+z_d)} = \sum_{n\geq 0}\binom{n(r_1+\cdots+r_d)}{nr_1,\ldots,nr_d}z_d^n.$$

Balanced word classes

$$\mathcal{L} = \{\text{binary expansions of } n \mid n \equiv 0 \mod 3.\}$$

Size = length of string

$$\mathcal{L} = (0 + (1(01^*0)^*1))^*$$

Parameter $\chi(w) = (|w|_0, |w|_1) = (\#0s \text{ in } w, \#1s \text{ in } w)$

$$\mathcal{L}_= = \{w \in \mathcal{L} \mid \#0s = \#1s\}$$

$$= \{\epsilon, 1001, 0011, 0110, 1100, 010101, 101010, 11100001, 10011001, \dots\}$$

$$L(x,y) = \frac{1}{1 - \left(x + \frac{y^2}{1 - \frac{x^2}{1-y}}\right)} \qquad L_=(y) = \Delta L(x,y)$$

$$L_=(y) = \Delta\left(1 + x + .. + y^2(1 + 2x + 4x^2 + ..) + y^3(x^2 + 2x^3 + 5x^4 + ..) + \dots\right)$$

$$= 1 + 4y^2 + 2y^3 + 36y^4 + \dots$$

(size by half length)

Other subseries extraction as diagonal

$F(\mathbf{z}, t)$ with series $\in K[[z_1, z_1^{-1}, \ldots, z_d, z_d^{-1}]][[t]]$:

$$\sum_{k\geq 0} \sum_{\mathrm{n}\in\mathbb{Z}^d} f(\mathbf{n}, k)\, \mathbf{z}^{\mathrm{n}} t^k$$

Constant Term

$$\begin{aligned}\mathrm{CT}\, F(\mathbf{z}, t) &= \sum_{n\geq 0} f(0, 0, \ldots, 0, n)\, t^n \\ &= \Delta F\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}, z_1 z_2 \ldots z_d\, t\right)\end{aligned}$$

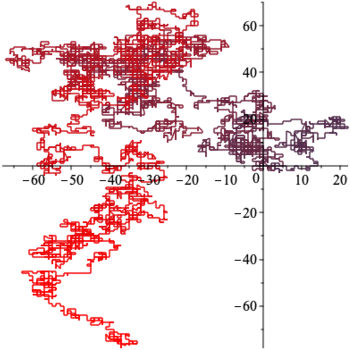
Positive Series

$$\begin{aligned}[z_1^{\geq 0} \ldots z_k^{\geq 0}] F(\mathbf{x}, t) &= \sum_{\mathrm{n}\in\mathbb{N}^d} f(\mathbf{n}, k)\, \mathbf{z}^{\mathrm{n}} t^k \\ &= \Delta\left(\frac{F\left(\frac{1}{z_1}, \ldots, \frac{1}{z_d}, z_1 z_2 \ldots z_d\, t\right)}{(1 - z_1) \ldots (1 - z_k)}\right)\end{aligned}$$

Excursions

Excursions: start and end at $(0,0)$ with steps from $S = \text{✥}$:

$$\mathcal{E} = \{w \in \{\uparrow, \downarrow, \leftarrow, \rightarrow\}^* \mid \chi(w) = (0,0)\}$$



OGF for excursions:

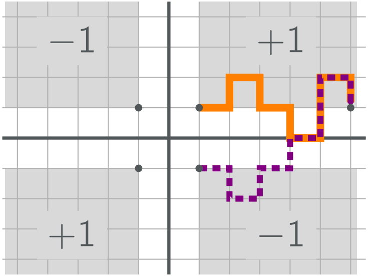
$$\sum walk_{\mathbb{Z}^2}^{\text{✥}}((0,0) \xrightarrow{n} (0,0))\, t^n = [x^0 y^0] \frac{1}{1 - t(x + 1/x + y + 1/y)}$$

$$= \Delta \frac{1}{1 - txy(1/x + x + 1/y + y))}$$

The set of combinatorial classes with OGF a diagonal of $\mathbb{N}$-rational is smaller than you'd like. (does not include Catalan!)
Differences of these classes are a wider class of series.

Walks confined to a quadrant - Reflection Principle

$$\sum_{n\geq 0} \text{walk}^{\leftrightarrow\updownarrow}_{\mathbb{N}^2}((0,0) \xrightarrow{n} (0,0))\, t^n$$



$$= [x^1y^1]\frac{xy - x/y + (xy)^{-1} + y/x}{(1 - t(x + 1/x + y + 1/y))}$$

$$= \mathrm{CT}\ \frac{\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)}{xy(1 - t(x + 1/x + y + 1/y))}$$

$$= \Delta\frac{xy\left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)}{1 - txy(x + 1/x + y + 1/y)}$$

$$= \Delta\frac{(x^2 - 1)(y^2 - 1)}{1 - t(x^2y + y + xy^2 + x)}.$$

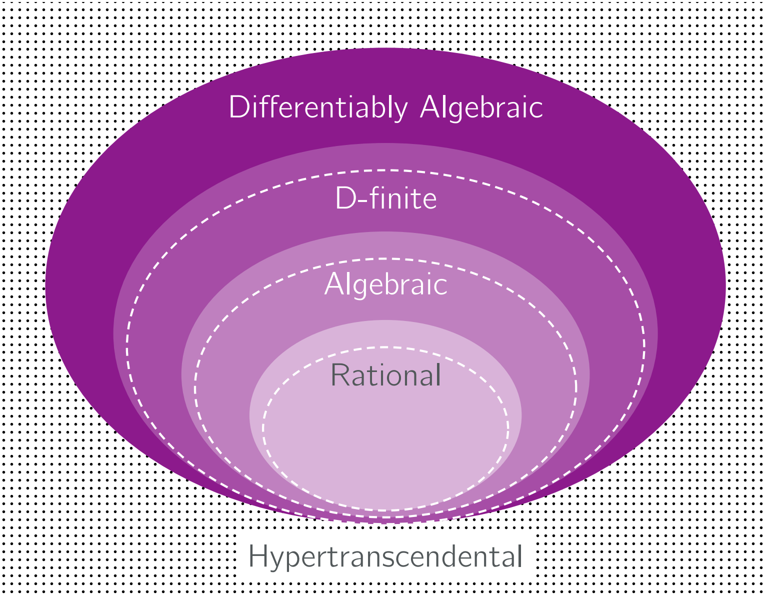
Diagonals and combinatorial generating functions

- Univariate algebraic functions are diagonals of bivariate rationals.
- D-finite functions
  - Algebraic functions are D-finite
  - Diagonals of D-finite functions are D-finite
  - OGFs of derived subclasses of algebraic and S-regular
  - Reflection principle – walks in Weyl chambers (from representation theory)

Lingering questions

Are combinatorial D-finite functions always diagonals? Are D-finite classes always (in bijection with) a derived class of an algebraic or regular? algebraic combination of derived classes?

Taxonomy of Generating Functions



Classic results of great utility to the combinatorialist

- Nature and type of singularities for series solutions of different equations types
- Behaviour near the singularities
- Asymptotic form of solutions for algebraic and linear ODE equations
- Results on series with positive coefficients (Pringsheim, Polya Carlson, Fatou,...)
  $F(z)$ converges inside the unit disc $\implies$ it is a rational function or transcendental over $\mathbb{Q}(z)$.

Transcendency

Transcendental OGF $\Longrightarrow$ class has no algebraic specification.

Trancendancy criterion

$$[z^n]F(z) \sim C\mu^n n^s, \quad s \notin \mathbb{Q} \setminus \{-1, -2, \dots\}$$

$$\mathcal{C} = \{u \in \{a, b, c\}^* \mid |u|_a \neq |u|_b \text{ or } |u|_a \neq |u|_c\}$$

$$\{a, b, c\}^* \setminus \mathcal{C} = \{u \in \{a, b, c\} \mid |u|_a = |u|_b = |u|_c\}$$

$$3^n - c_n = \binom{3n}{n, n, n}$$

$$\underbrace{\sum 3^n z^n}_{\text{rational}} - \sum c_n z^n = \underbrace{\sum \underbrace{\binom{3n}{n, n, n}}_{\sim C\, 27^n n^{-1}} z^n}_{\textit{transcendental}}$$

$$\Longrightarrow \sum c_n z^n \text{ transcendental}$$

Conclusion

1. Dictionary between combinatorial specification and OGF functional equations
2. 3 families of combinatorial classes: S-regular, algebraic, derived subclasses
3. Use results on the nature of solutions to help sort objects and make effective computation
4. Diagonal operator is used to describe many combinatorial classes

$$\Delta F(\mathbf{z}, t) = \Delta \sum_{k \geq 0} \sum_{\mathrm{n} \in \mathbb{Z}^d} f(\mathbf{n}, k)\, \mathbf{z}^{\mathrm{n}} t^k := \sum_{n \geq 0} f(n, n, \ldots, n)\, t^n.$$

5. Christol's Conjecture: Every G-series is a diagonal of a rational function
6. Next: Given a multivariable rational function, determine the coefficient asymptotics of a diagonal.

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II. Singularities and Critical Points

Objective

Systematic methods to determine asymptotic estimates for the number of objects of size n in combinatorial class $\mathcal{C}$.

Understand the singularity structure of its OGF $C(z)$

Part II: Focus on diagonals of multivariable rationals (e.g. OGFs of derived classes)

$$\Delta \frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} = \Delta \sum_{k \geq 0} \sum_{\mathrm{n} \in \mathbb{Z}^d} f(\mathbf{n}, k) \, \mathbf{z}^{\mathrm{n}} t^k := \sum_{n \geq 0} f(n, n, \ldots, n) \, t^n.$$

Analytic Strategy

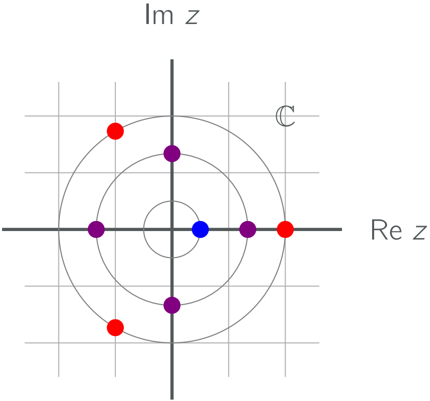
- Relate the singularities of $\frac{G(\mathrm{z},t)}{H(\mathrm{z},t)}$ and $\Delta \frac{G(\mathrm{z},t)}{H(\mathrm{z},t)}$.
- Study the geometry of the variety of points annihilating H.

Univariate Singularity

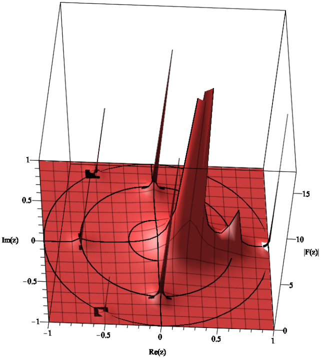
Poles

$$F(z) = \frac{1}{(1-z^3)(1-4z)^2(1-5z^4)}$$

$1 + 8z + 54\,z^2 + 304\,z^3 + 1599\,z^4 + 7928\,z^5 + O\left(z^6\right)$



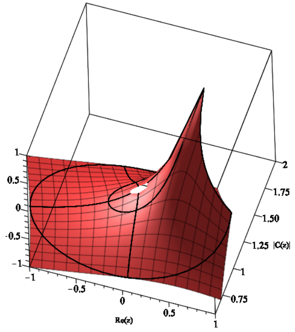
Poles of $F(z)$



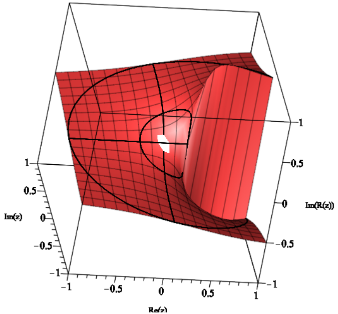
Value of $|F(z)|$

Branch Point Singularity

$$C(z) = \frac{1-\sqrt{1-4z}}{2z} \quad 1+z+2z^2+5z^3+14z^4+42z^5+132z^6+O(z^7)$$



$|C(z)|$



$\operatorname{Im} C(z)$

Exponential growth and the radius of convergence

$$F(z) = \sum_{n \geq 0} f_n z^n \in \mathbb{R}_{>0}[[z]]$$

$\implies$ there is a positive real valued singular point $\rho \in \mathbb{R}_{>0}$ on the circle of convergence. (Pringshheim)

Exponential growth

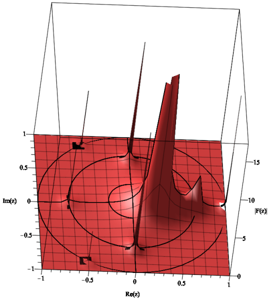
$$\mu := \limsup_{n \to \infty} f_n^{1/n}$$

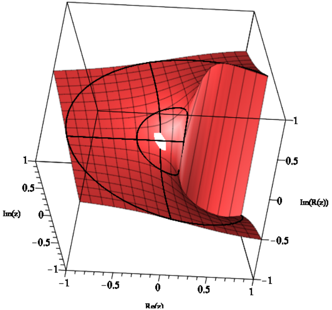
$$\text{“}f_n \sim \kappa \mu^n n^\alpha\text{”}$$

$$ROC(F(z)) = \rho \implies \mu = \rho^{-1}$$

Convergence and the Exponential Growth

The radius of convergence $F(z) = \rho$
$\implies$ exponential growth of $[z^n]F(z) = \rho^{-1}$.





$$\frac{1}{(1-z^3)(1-4z)^2(1-5z^4)}$$

$$\lim_{n\to\infty} r_n^{1/n} = 4$$

$$\frac{1-\sqrt{1-4z}}{2z}$$

$$\lim_{n\to\infty} c_n^{1/n} = 4$$

The first principle of coefficient asymptotics

The **location** of singularities of an analytic function determines the **exponential order** of growth of its Taylor coefficients.

We connect the boundary of convergence and exponential growth.

Preview: An analogy

Here is a rough idea of what the multivariable case looks like.

Univariate Rationals

$$F(z) = \frac{G(z)}{H(z)} = \sum f_n z^n$$

$$f_n \sim C\mu^n n^\alpha$$

dominant singularity: $\rho \in \mathbb{C}$ on circle of convergence satisfying $H(\rho) = 0$

$$\mu = |\rho|^{-1}$$

Multivariable Rationals

$$\Delta \frac{G(z_1, \ldots, z_d)}{H(z_1, \ldots, z_d)} = \sum f_{nn\ldots n} z_d^n$$

$$f_{nn\ldots n} \sim C\mu^n n^\alpha$$

minimal critical point: $(\rho_1, \ldots, \rho_d)$ on the boundary of convergence satisfying $H(\rho_1 \ldots, \rho_d) = 0$ + other equations.

$$\mu = |\rho_1 \ldots \rho_d|^{-1}$$

Multivariable Series

Convergence of multivariable series

- View the series as an iterated sum.

$$\sum_{n_d}\left(\dots\left(\sum_{n_1} a(n_1,\dots,n_d)z_1^{n_1}\right)\dots\right)z_d^{n_d}$$

- The domain of convergence, denoted $\mathcal{D} \subseteq \mathbb{C}^d$, is the interior of the set of points where the series converges absolutely.
- The polydisk of a point z is the domain

$$D(\mathbf{z}) = \{\mathbf{z}' \in \mathbb{C}^d : |z_i'| \leq |z_i|, 1 \leq i \leq d\}.$$

- The torus associated to a point is

$$T(\mathbf{z}) = \{\mathbf{z}' \in \mathbb{C}^d : |z_i'| = |z_i|, 1 \leq i \leq d\}.$$

- A domain of convergence is multicircular.

$$\mathbf{z} = (z_1,\dots,z_d) \in \mathcal{D} \implies T(\mathbf{z}) \subseteq \mathcal{D} \implies (\omega_1 z_1,\dots,\omega_d z_d) \in \mathcal{D}, \quad |\omega_k| = 1$$

What is a singularity of G/H?

The set of singularities of $G(z)/H(z)$ is the algebraic variety

$$\mathcal{V} := \{\mathbf{z} : H(\mathbf{z}) = 0\}.$$

minimal points (working definition)

The set of minimal points of a series expansion of F is the set of singular points on the boundary of convergence.

$$\mathcal{V} \cap \partial\mathcal{D}$$

A point $\mathbf{z}$ is **strictly minimal** if $\mathcal{V} \cap D(\mathbf{z}) = \{\mathbf{z}\}$

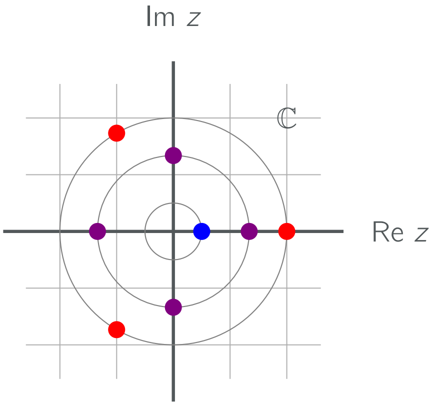
Example: 1D

Definitions

The set of minimal points of a series development of F is the set of singular points on the boundary of convergence.

$$\mathcal{V} \cap \partial\mathcal{D}$$

A point $\mathbf{z}$ is strictly minimal if $\mathcal{V} \cap D(\mathbf{z}) = \{\mathbf{z}\}$



$$H(z) = (1 - z^3)(1 - 4z)^2(1 - 5z^4)$$

$\partial\mathcal{D} = \{z : |z| = 1/4\}$, $\mathcal{V} = \{1/4, 1, w, w^2, (\frac{1}{5})^{1/4}\}$
minimal point: $\mathcal{V} \cap \mathcal{D} = \{1/4\}$, strictly minimal.

Example: $F(x,y) = \frac{1}{1-x-y}$

Taylor expansion: $\sum_{k,\ell} \binom{k+\ell}{k} x^k y^\ell$

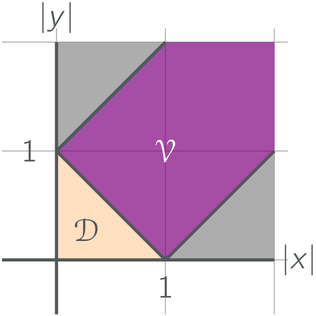
Convergence at (x,y)

$\implies$ convergence at $(|x|,|y|)$

$$F(|x|,|y|) = \frac{1}{1-|x|-|y|} \implies |x|+|y|<1$$

$$\partial\mathcal{D} = \{(x,y)\in\mathbb{C}^2 \mid |x|+|y|=1\}$$

$$\mathcal{V} = \{(z,1-z) \mid z\in\mathbb{C}\}$$



Minimal points $\mathcal{V}\cap\partial\mathcal{D}$

$$\{(z,1-z) \mid |z|+|1-z|=1\} = \{(x,1-x) \mid x\in\mathbb{R}_{>0}\}$$

All strictly minimal.

A first formula for exponential growth for diagonal coefficients

Convergence and exponential growth

Given series $\sum a(\mathbf{n})\,\mathbf{z}^{\mathrm{n}}$ and $\mathbf{z} \in \mathcal{D}$,

$$\sum_{\mathrm{n}\in\mathbb{N}^d} a(\mathbf{n})|z_1|^{n_1}|z_2|^{n_2}\dots|z_d|^{n_d} \text{ is convergent (absolute conv).}$$

$$\Longrightarrow \sum_{n\in\mathbb{N}} a(n,n,\dots,n)\,|z_1|^n|z_2|^n\dots|z_d|^n \text{ is convergent (subseries).}$$

$$= \sum_n a(n,n,\dots,n)|z_1 z_2 \dots z_d|^n$$

$\Longrightarrow$ $t = |z_1 z_2 \dots z_d|$ is within the radius of convergence of $\Delta F(\mathbf{z})$.

$$\mu \le \limsup_{n\to\infty} |a(n,n,\dots,n)|^{1/n} \le |z_1 z_2 \dots z_d|^{-1} \quad \text{with } \forall \mathbf{z} \in \overline{\mathcal{D}}$$

$$\le \inf_{(z_1,\dots,z_d)\in\overline{\mathcal{D}}} |z_1 z_2 \dots z_d|^{-1}.$$

Thm: Under conditions of non-triviality, the infimum is reached at a minimal point:

$$\mu = \inf_{\mathrm{z}\in\partial\mathcal{D}\cap\mathcal{V}} |z_1 \dots z_d|^{-1}. \tag{5}$$

Example: Binomials $F(x, y) = (1 - x - y)^{-1}$

Minimal points: $\partial\mathcal{D} \cap \mathcal{V} = \{(x, 1-x) \in \mathbb{R}^2 : 0 < x < 1\}$.

$\mu = \limsup_{n\to\infty} a(n,n)^{1/n} = \inf_{(x,y)\in\partial\mathcal{D}\cap\mathcal{V}} |xy|^{-1} = \inf_{x\in\mathbb{R}:0\leq x\leq 1} (x(1-x))^{-1} = 4.$

We can consider non-central diagonals.

$$\limsup_{n\to\infty} a_{rn\,sn}^{1/n} = \inf_{(x,y)\in\partial\mathcal{D}} |x^r y^s|^{-1} = \inf_{x\in\mathbb{R}} (x^r(1-x)^s)^{-1}.$$

This is minimized at $x = \frac{r}{r+s}$.
The exponential growth:

$$\mu = \left(\left(\frac{r}{r+s}\right)^r \left(\frac{s}{r+s}\right)^s\right)^{-1}.$$

We got lucky here – we could easily write y in terms of x. What to do in general?

Computing critical points

The height function h

Astuce

We convert the multiplicative minimization to a linear minimization using logarithms.

To minimize $|z_1 \dots z_d|^{-1}$, minimize:

$$-\log|z_1 \dots z_d| = \underbrace{-\log|z_1| - \cdots - \log|z_d|}_{\text{linear in } \log|z_i|}$$

Define a function $h : \mathcal{V}^* \to \mathbb{R}$:

$\mathcal{V}^* = \mathcal{V} \setminus \{z : z_1 \dots z_d \neq 0\}$

$$(z_1, \dots, z_d) \mapsto -\log|z_1| - \cdots - \log|z_d|.$$

The map h is smooth $\implies$ minimized at its critical points. When $r = (1, \dots, 1)$, the critical points are solutions to the critical point equations:

$$H(\mathbf{z}) = 0, \quad z_1 \frac{\partial H(\mathbf{z})}{\partial z_1} = z_j \frac{\partial H(\mathbf{z})}{\partial z_j}, 2 \leq j \leq d.$$

Critical points

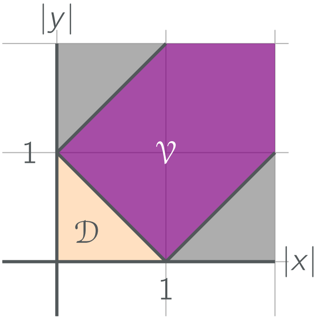
Critical points are potential locations of minimizers of $|z_1 \dots z_d|^{-1}$.
In the most straightforward cases it suffices to compare the values of this product and select the critical point that is the global minimizer.

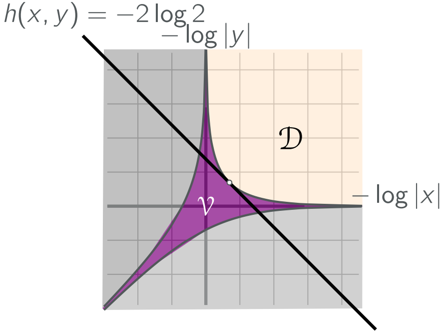
- A critical point is strictly minimal if it is on the boundary of convergence of the series.
- In these generating functions the asymptotics is driven by a finite number of isolated minimal points. Simplest case.

Visualize Critical Points

Critical points of $(1 - x - y)^{-1}$ for $r = (1, 1)$

1. $\rho \in \mathcal{V} \cap \partial\mathcal{D}$ — $\rho = (1/2, 1/2)$
2. minimize $h(x, y) = -\log|x| - \log|y|$ — $h(x, y) = -2\log 2$
3. $\mu = |\rho_1 \rho_2|^{-1}$ — $\lim_{n\to\infty} \binom{2n}{n}^{1/n} = 4$





$(x, y) \mapsto (-\log|x|, -\log|y|)$

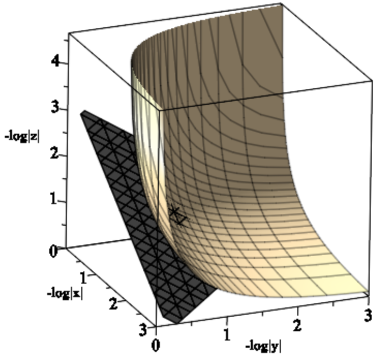
Trinomial $(1-x-y-z)^{-1}$

Critical points

1. $\rho \in \mathcal{V} \cap \partial\mathcal{D}$ $\qquad \rho = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
2. minimize $h(x,y,z) = -\log|x| - \log|y| - \log|z|$

$$h(x,y,z) = 3\log 3$$

3. $\mu = |\rho_1\rho_2\rho_3|^{-1}$ $\qquad \lim_{n\to\infty} \binom{3n}{n,n,n}^{1/n} = 27$



Non-central diagonals

If we want a non-central diagonal, we want to minimize

$$|z_1^{r_1} \dots z_d^{r_d}|^{-1} \quad in \quad \partial\mathcal{D} \cap \mathcal{V}.$$

Instead take height function here is

$$(z_1, \dots, z_d) \mapsto -r_1 \log|z_1| - \cdots - r_d \log|z_d|.$$

The equations change. For example, in 2D, diagonal (r, s), solve the equations:

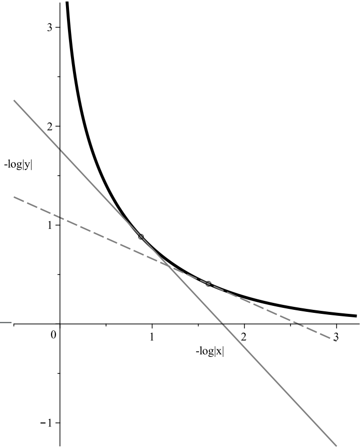
$$H(x, y) = 0, \quad s\, x \frac{\partial H(x, y)}{\partial x} = r\, y \frac{\partial H(x, y)}{\partial y}.$$

Critical point depends on the diagonal ray

Delannoy Numbers

$d(rn, sn) := [x^{rn}y^{sn}](x + y + xy)^n$

(r,s)	$-r\log\lvert x\rvert - s\log\lvert y\rvert$	ρ
$(1,1)$	———	$\left(\frac{1}{\sqrt{2}-1}, \frac{1}{\sqrt{2}-1}\right)$
$(5,2)$	– – –	$\left(\frac{1}{5}, \frac{2}{3}\right)$



Summary: To Find Critical Points

Given: $G(x, y)/H(x, y) = \sum f_{k,\ell}\, x^j y^k$, irreducible H
$(r, s) \in \mathbb{R}^2_{>0}$
Determine: $\mu = \limsup_{n\to\infty} f_{rn,sn}^{1/n}$, critical points ρ

- Find solutions $\{\rho\}$ to the (r, s)-critical point equations.
  Hint: Find Gröbner basis of
  ```
  [H, s*x*diff(H, x)-r*y*diff(H,y)]
  ```
- Ensure $T(\rho) \subset \partial\mathcal{D}$
- Set $\mu = \min |\rho_1 \ldots \rho_d|^{-1}$ among those solutions with no 0 coordinate.
- We use the set of such ρ to find the sub-exponential growth (tomorrow)
- Nontriviality requirement: ρ to be smooth as *a function of* (r, s) near where you want it.

Balanced Binary Words

Let $\mathcal{L}$ =Binary words over $\{0,1\}$ with no run of 1s of length 3.

$$\mathcal{L} = (\epsilon + 1 + 11) \cdot (0 \cdot (\epsilon + 1 + 11))^*$$

Parameter: $\chi(w) = (|w|_0\,, |w|_1)$

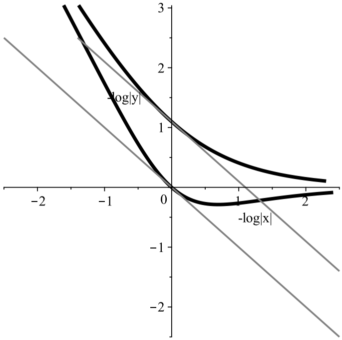
$$\mathcal{L}_= = \{w \in \mathcal{L} \mid \chi(w) = (n,n)\}$$

$$L_=(y) = \Delta \frac{1 + x + x^2}{1 - y(1 + x + x^2)}$$

- GB of Critical point equations: $[x^2 - 1, x + 3y - 2]$
- two solutions: $(1, 1/3)$ $(-1, 1)$
- $\mu = \min |\rho_1 \ldots \rho_d|^{-1}$ $(1, 1/3) \mapsto 3$ $(-1, 1) \mapsto 1$
- BUT $(1,1) \in T(-1,1) \implies (-1,1)$ is not a minimal point because $(1,1)$ outside of domain of convergence.

$$[y^n]L_=(y) \to \kappa 3^n n^\alpha$$

Visualize the boundary



Simple Excursions

Let $\mathcal{E}$ be the set of simple excursions in the entire plane, that is walks that start and end at the origin, taking unit steps $\{\uparrow, \downarrow, \leftarrow, \rightarrow\}$

$$e(n) = [x^0 y^0]\,(x + 1/x + y + 1/y)^n$$

We can deduce:

$$E(z) = \Delta \frac{1}{1 - zxy\left(x + \frac{1}{x} + y + \frac{1}{x}\right)}$$

Any critical point $\rho = (x, y, z)$ will have $z = \frac{1}{xy(x+1/x+y+1/y)}$ from $H = 0$.
Critical points: $(1, 1, 1/4), (-1, -1, -1/4)$

$$e(2n)^{1/2n} = \inf_{\rho \in \partial\mathcal{D} \cap \mathcal{V}} |xyz|^{-1} = \inf_{0 \leq x, y \leq 1} |x + 1/x + y + 1/y| = 4^2$$

Excursions for any finite step set

This phenomena is general. Let $\mathcal{S}$ be any weighted finite 2D step set

$$S(x,y) = \sum_{(j,k)\in\mathcal{S}} w(j,k)\, x^j y^k$$

$$e(n) = [x^0 y^0] S(x,y)^n$$

We can deduce:

$$E(z) = \Delta \frac{1}{1 - zxyS(1/x, 1/y)}$$

Any critical point $\rho = (x,y,z)$ will have $z = \frac{1}{xyS(1/x,1/y)}$ from $H = 0$.

$$\limsup_{n\to\infty} e(n)^{\frac{1}{n}} = \inf_{\rho\in\partial\mathcal{D}\cap\mathcal{V}} |xyz|^{-1} = \inf_{\rho\in\partial\mathcal{D}} |S(1/x,1/y)|$$

The minimum is found using the critical point eqn.

What if H factors?

Suppose H factors nontrivially into squarefree factors:

$$H = H_1 \dots H_k$$

- CASE A: $H_j(\rho) = 0 \implies H_k(\rho) \neq 0$ for $j \neq k$: OK.
- CASE B: Must decompose $\mathcal{V}$ into strata, and find critical points for each stratum independently. **Important to keep track of the co-dimension of the stratum for later.**

Walks in the quarter plane that end anywhere

Let $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$. $\mathcal{T}$= walks start at $(0,0)$ end anywhere. Using a reflection principle argument:

$$T(z) = \Delta \frac{\left(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y\right)}{\left(1 - zxy(1/x + x/y + y)\right)(1-x)(1-y)} \tag{6}$$

Critical points

We divide $\mathcal{V}_H$ into strata and we determine critical points from each of them.

Image of $\mathcal{V}$ under $(x, y, z) \mapsto (|x|, |y|, |z|)$

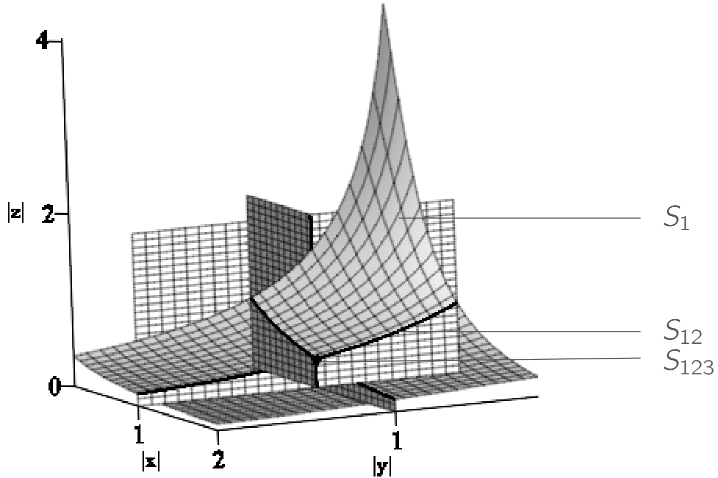
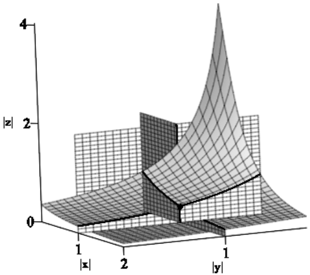
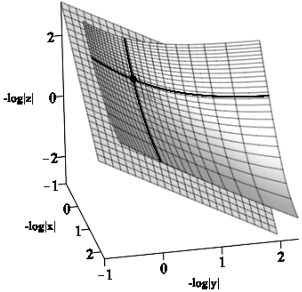


Image of $\mathcal{V}$ under $(x, y, z) \mapsto (-\log|x|, -\log|y|, -\log|z|)$





Stratum	Critical points	value of $\lvert xyz\rvert^{-1}$
S_1	$(w^2, w, w/3)$, $(w, w^2, w^2/3)$	$1/3$
S_{12}		
S_{23}		
S_{123}	$(1, 1, 1/3)$	$1/3$

A lattice path enumeration problem

Let $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$. $\mathcal{T}$= walks start at $(0,0)$ end anywhere. Using a reflection principle argument:

$$T(z) = \Delta \frac{\left(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y\right)}{(1 - zxy(1/x + x/y + y))\,(1-x)(1-y)}$$

We conclude: Three critical points:

$$(w, w^2, w^2/3), (w^2, w, w/3), (1, 1, 1/3)$$

(Potential for periodicity..)

Exponential growth: $t_n \sim C 3^n n^\alpha$.
Tomorrow: find C, α.

Next..

Determine **how** each contributing critical point modulates the dominant exponential term by a subexponential factor.

Bibliography

Analytic Combinatorics in Several Variables
Pemantle and Wilson 2013

III. Diagonal Asymptotics

The problem

Diagonal Asymptotics

Given:

$$F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum f(\mathbf{n})\mathbf{z}^{\mathrm{n}}$$

Determine the asymptotics of $f(n, n, \ldots, n)$ as $n \to \infty$

- Singular Variety $\mathcal{V} = \{\mathbf{z} \mid H(\mathbf{z}) = 0\}$
- Minimal Points: $\partial\mathcal{D} \cap \mathcal{V}$
- Critical points minimize: $|\rho_1 \ldots \rho_d|^{-1}$ (with value μ, say)
- Minimal critical point ρ contained in both

$$\underbrace{-\log|z_1| - \cdots - \log|z_d| = \log\mu}_{\text{a hyperplane}} \qquad \rho \in \partial\mathcal{D} \cap \mathcal{V}$$

Balanced Binary Words

Let $\mathcal{L}$ =Binary words over $\{0,1\}$ with no run of 1s of length 3.

$$\mathcal{L} = (\epsilon + 1 + 11) \cdot (0 \cdot (\epsilon + 1 + 11))^*$$

Parameter: $\chi(w) = (|w|_0, |w|_1)$

$$\mathcal{L}_= = \{w \in \mathcal{L} \mid \chi(w) = (n, n)\}$$

$$L_=(y) = \Delta \frac{1 + x + x^2}{1 - y(1 + x + x^2)}$$

- GB of Critical point equations: $[x^2 - 1, x + 3y - 2]$
- two solutions: $(1, 1/3)$ $(-1, 1)$
- $\mu = \min |\rho_1 \ldots \rho_d|^{-1}$ $(1, 1/3) \mapsto 3$ $\qquad (-1, 1) \mapsto 1$
- BUT $(1, 1) \in T(-1, 1) \implies (-1, 1)$ is not a minimal point because $(1, 1)$ outside of domain of convergence.

$$[y^n]L_=(y) \to \kappa 3^n n^\alpha$$

First Principle of Coefficient Asymptotics

The **location** of singularities of an analytic function determines the **exponential order** of growth of its Taylor coefficients.

We connect the boundary of convergence and exponential growth.

Second Principle of Coefficient Asymptotics

The **nature** of the singularities determines the way the dominant exponential term in coefficients is modulated by a subexponential factor.

Nature = geometry of the singular variety at the critical point.

The problem

Diagonal Asymptotics

Given:

$$F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum f(\mathbf{n})\mathbf{z}^{\mathrm{n}}$$

Determine the asymptotics of $f(n, n, \ldots, n)$ as $n \to \infty$

- Singular Variety $\mathcal{V} = \{\mathbf{z} \mid H(\mathbf{z}) = 0\}$
- Minimal Points: $\partial\mathcal{D} \cap \mathcal{V}$
- Critical points minimize: $|\rho_1 \ldots \rho_d|^{-1}$ (with value μ, say)
- Minimal critical point ρ satisfies

$$\underbrace{-\log|\rho_1| - \cdots - \log|\rho_d| = \log\mu}_{\text{a hyperplane}} \qquad \rho \in \partial\mathcal{D} \cap \mathcal{V}$$

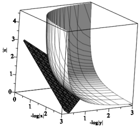
What are the geometries we can handle?

Behaviour at critical point ρ

$$-\log|\rho_1| - \cdots - \log|\rho_d| = \log\mu \quad \rho \in \partial\mathcal{D} \cap \mathcal{V}$$

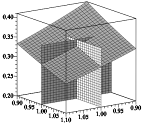
Smooth Point THM A (Hörmander)

e.g. excursions



Transverse Multiple Point THM B (Pemantle and Wilson)

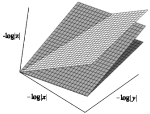
e.g. general walks



Arrangement Point Decompose F;THM B

e.g. vector partitions

polytope dilation



Univariate Case

Cauchy Integral Formula

The heart of analytic combinatorics.

Theorem

Suppose that $F(z)$ is analytic at the origin, with Taylor series expansion $F(z) = \sum_{n \geq 0} f(n)\, z^n$. Then for all $n \geq 0$,

$$f(n) = \frac{1}{(2\pi i)} \int_\gamma \frac{F(z)}{z^{n+1}}\, dz \tag{7}$$

where γ is a counterclockwise circle about the origin sufficiently small that $F(z)$ is analytic in its interior and is continuous on it.

Strategies

1. Estimate integral by curvelength $\times$ max on curve
2. Use someone else's estimate.

Transfer Theorem

Theorem

Assume that, with the sole exception of the singularity $z = 1$, $F(z)$ is analytic in the domain

$$\Omega = \{z \mid |z| \leq 1 + \nu, \ |\arg(z-1)| \geq \phi\}$$

for some $\nu > 0$ and $0 < \phi < \pi/2$. Assume further that as z tends to 1 in Ω,

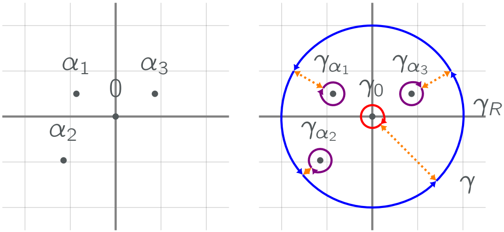
$$F(z) = (1-z)^{\alpha} \log\left(\frac{1}{1-z}\right)^{\beta} O\left(\left(\log \frac{1}{1-z}\right)^{-1}\right)$$

for some real α and β such that $\alpha \notin \{0, 1, 2, \ldots\}$. Then

$$[z^n]F(z) = \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \log^{\beta} n \left(O\left(\log^{-1} n\right)\right).$$

Ref: Flajolet Odlyzko, 1990

Rewrite as a sum of residues



$$
\begin{aligned}
0 &= \int_\gamma F(z)/z^{n+1} dz \\
&= \int_{\gamma_R} F(z)/z^{n+1} dz + \underbrace{\int_{\gamma_0} F(z)/z^{n+1} dz}_{f(n)} + \sum_{s \in \{\alpha_1, \ldots, \alpha_m\}} \int_{\gamma_s} F(z)/z^{n+1}. \\
&= O(R^{-n}) + f(n) + \sum_{s \in \{\alpha_1, \ldots, \alpha_m\}} \int_{\gamma_s} F(z)/z^{n+1}.
\end{aligned}
$$

$$f(n) = \sum_{s \in \{\alpha_1, \ldots, \alpha_m\}} \int_{\gamma_s} F(z)/z^{n+1} + O(R^{-n})$$

Key Multivariable Theorem

Multidimensional Cauchy Integrals

Theorem

Fix d and let $\mathbf{z} = (z_1, \ldots, z_d)$. Suppose that $F(\mathbf{z}) \in \mathbb{C}(\mathbf{z})$ is analytic at the origin, with Taylor series expansion $F(\mathbf{z}) = \sum_{\mathrm{n}\in\mathbb{N}^d} f(\mathbf{n})\,\mathbf{z}^{\mathrm{n}}$ Then for all $n \geq 0$,

$$f(\mathbf{n}) = \frac{1}{(2\pi i)^d} \int_T \frac{F(\mathbf{z})}{(z_1 \ldots z_d)^n} \cdot \frac{dz_1\ \ldots,\ dz_d}{z_1 \ldots z_d},$$

where T is the torus $T(\epsilon) = T(\epsilon_1, \epsilon_2, \ldots, \epsilon_d)$ has each ϵ_j sufficiently small such that $F(\mathbf{z})$ is analytic in the interior of $D(\epsilon)$, and is analytic on the boundary.

Smooth Point Asymptotics

Case 1: Smooth point asymptotics

The minimal critical point ρ is a smooth point if $\partial_k H(\rho) \neq 0$ for all k.

Example

The point $(1, 1, 1/3)$ is a smooth critical point for $H(x, y) = 1 - y(1 + x + x^2)$:

Compute $H_x(x, y) = y(1 + 2x), \quad H_y = (1 + x + x^2)$

Verify $H_x(1, 1, 1/3) = 1 \neq 0, H_y(1, 1, 1/3) = 3 \neq 0.$

Tool: Fourier Laplace Integrals

The strategy is to rewrite Cauchy Integrals as Fourier-Laplace integrals:

$$\int_{\mathcal{N}} A(\mathbf{t}) e^{-\lambda \phi(\mathrm{t})} dt_1 \dots dt_d,$$

with the functions A and ϕ analytic over their domain of integration, and $\mathcal{N}$ is some neighbourhood in $\mathbb{R}^d$.
As we saw yesterday, often asymptotic estimate for these integrals are known.

Smooth Point: THM A

Suppose $A : \mathbb{C}^d \to \mathbb{C}$ and $\phi : \mathbb{C}^d \to \mathbb{C}$ are both smooth in a neighbourhood $\mathcal{N}$ of $\mathbf{0}$ and that

- $\phi(\mathbf{0}) = 0$
- ϕ has a critical point at $\mathbf{t} = \mathbf{0}$, i.e., that $(\nabla\phi)(\mathbf{0}) = \mathbf{0}$, and that the origin is the only critical point of ϕ in $\mathcal{N}$;
- the Hessian matrix $\mathcal{H}$ of ϕ has i, j^{th} entry $\partial_i \partial_j \phi(\mathbf{t})$, and at $\mathbf{t} = \mathbf{0}$ is non-singular;
- the real part of $\phi(\mathbf{t})$ is non-negative on $\mathcal{N}$.

Then for any integer $M > 0$ there exist computable constants $C_0, \ldots, C_M$ such that

$$\int_{\mathcal{N}} A(\mathbf{t})\, e^{-n\phi(\mathrm{t})} d\mathbf{t} = \left(\frac{2\pi}{n}\right)^{d/2} \det(\mathcal{H})^{-1/2} \cdot \sum_{k=0}^{M} C_k n^{-k} + O\left(n^{-M-1}\right).$$

$C_0 = A(\mathbf{0})$

If $A(\mathbf{t})$ vanishes to order L at 0 then $C_0 = \cdots = C_{\lfloor \frac{L}{2} \rfloor} = 0$. [Hörmander]

Balanced Binary Words with no Runs

We complete the asymptotic analysis of the number of balanced binary words over $\{0, 1\}$ such that no word has a run of 1s of length 3 or longer. The generating function by halflength is

$$\Delta \frac{1 + x + x^2}{1 - y(1 + x + x^2)}.$$

There is a minimal critical point at $\rho = (1, 1/3)$.

Strategy

1. Remark $[x^n y^n](1 + x + x^2)(y(1 + x + x^2))^n = [x^n](1 + x + x^2)^{n+1}$
2. Write Cauchy Integral in one smaller dimension
3. Rewrite as a sum of integrals around the singularities
4. For each integral, move the singularity to 0 in a way that converts it to a Fourier-Laplace integral.

Rewriting a Cauchy Integral as Fourier-Laplace Integral

$[x^n](1 + x + x^2)^{n+1}$

$$
\begin{aligned}
[x^n]A(x)B(x)^n &= \frac{1}{2\pi i}\int_{|x|=\epsilon} \frac{A(x)B(x)^n}{x^{n+1}}\,dx \\
&= \frac{1}{2\pi i}\int_{|x-\rho|=\epsilon} \frac{A(x)B(x)^n}{x^{n+1}}\,dx + O((\rho+\epsilon)^{-1}) \\
&= \frac{1}{2\pi i}\int_{\mathcal{N}} \frac{A(\rho e^{it})B(\rho e^{it})^n}{\rho^{n+1}e^{it(n+1)}} i\rho e^{it\rho}\,dt \\
&= \frac{\rho^{-n}B(\rho)^n}{2\pi}\int_{\mathcal{N}} A(\rho e^{it})\frac{B(\rho e^{it})^n}{B(\rho)^n}e^{-it(n+1)}\,dt \\
&= \frac{\rho^{-n}B(\rho)^n}{2\pi}\int_{\mathcal{N}} A(\rho e^{it})e^{-n\phi(t)}\,dt
\end{aligned}
$$

with $\phi(t) = \log\frac{B(\rho)}{B(\rho e^{it})} + it$. $A : \mathbb{C}^d \to \mathbb{C}$ and $\phi : \mathbb{C}^d \to \mathbb{C}$ are both smooth in a neighbourhood $\mathcal{N}$ of $\mathbf{0}$; ϕ has a critical point at $\mathbf{t} = \mathbf{0}$, the origin is the only critical point of ϕ in $\mathcal{N}$;

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$$3\frac{3^n}{\sqrt{2\pi n}}$$

How to apply to more general cases

The first step of the computation used

$$H = 1 - yB(x) \implies y = B(x)^{-1} \text{ on } \mathcal{V}$$

If the variety $\mathcal{V}$ is smooth, and if $\partial_{d+1}H(\rho) \neq 0$, then by the implicit function theorem there is a parametrization $z_d = \Psi(z_1, \ldots, z_{d-1})$ that we can similarly use.
We define ϕ and proceed as above:

$$\begin{aligned}\phi(\mathbf{t}) = \log(\psi(\rho_1 e^{it_1}, \ldots, \rho_{d-1} e^{it_{d-1}})) - \log(\rho) \\ + \frac{i}{r_d}\left(r_1 t_1 + \cdots + r_{d-1} t_{d-1}\right).\end{aligned}$$

A prefabricated theorem for 2D

Theorem (Pemantle + Wilson)

Let $G(x, y)/H(x, y)$ be meromorphic and suppose that as $\hat{\mathbf{r}}$ varies in a neighbourhood N of (r, s), there is a smoothly varying, strictly minimal smooth critical point ρ in the direction (r, s). Finally $G(\rho) \neq 0$. Define $\mathbf{z}(r, s)$ as the critical point in the direction (r, s), and define

$$Q(\mathbf{z}(\hat{\mathbf{r}})) := -y^2 H_y^2 x H_x - y H_y x^2 H_x - x^2 y^2 \left(H_y^2 H_{xx} + H_x^2 H_y y - 2 H_x H_y H_{xy}\right).$$

If this function is nonzero in a neighbourhood of (rn, sn) then

$$f(rn, sn) \sim \frac{G(\rho)\left(x^{-r} y^{-s}\right)^n}{\sqrt{2\pi}} \sqrt{\frac{-\rho_2 H_y(\rho)}{nsQ(\rho)}}.$$

Transversal Intersections

Transversal Intersection

Intuitively, two curves have a transversal intersection if the intersection is robust to small perturbations of the curves.

- YES Two non-parallel lines in $\mathbb{R}^2$
- NO Two non-parallel lines in $\mathbb{R}^3$
- NO $y = x^2$ and $y = 0$ at $(0,0)$
- NO Three lines in $\mathbb{R}^2$

Proposition

Point $\rho \in \mathcal{V}$ is a multiple point if and only if there is a factorization $H = \prod_{j=1}^{N} H_j^{m_j}$ with $\nabla H_j(\rho) \neq 0$ and $H_j(\rho) = 0$.
Point ρ is a transverse multiple point of order N if in addition the gradient vectors are linearly independent.

A lattice path enumeration problem

Let $\mathcal{S} = \{\nwarrow, \rightarrow, \downarrow\}$. $\mathcal{T}$= walks start at $(0,0)$ end anywhere. Using a reflection principle argument:

$$T(z) = \Delta \frac{\left(1 - y^2/x + y^3 - x^2y^2 + x^3 - x^2/y\right)}{(1 - zxy(1/x + x/y + y))\,(1-x)(1-y)}$$

Three critical points:

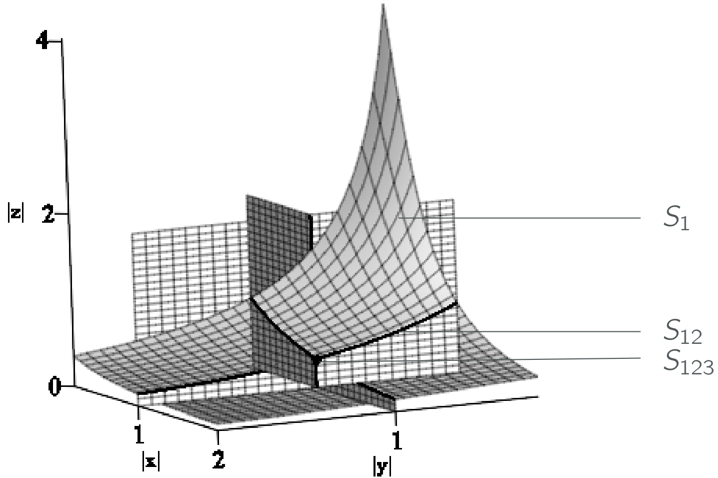
$$(w, w^2, w^2/3), (w^2, w, w/3), (1, 1, 1/3)$$

The point $(1,1,1/3)$ is at the intersection of more than one variety. Is it transversal?

$$\begin{bmatrix} \nabla H_1(1,1,1/3) \\ \nabla H_2(1,1,1/3) \\ \nabla H_3(1,1,1/3) \end{bmatrix} = \begin{bmatrix} -1 & -1 & -3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

This matrix of full rank, and so YES $(1,1,1/3)$ is a transversal multiple point of order 3.

Image of $\mathcal{V}$ under $(x, y, z) \mapsto (|x|, |y|, |z|)$



Transversal Intersection: THM B

Pemantle and Wilson 2013

Theorem 10.3.3 (complete intersection) *Let* $F = G / \prod_{j=1}^{d} H_j^{m_j}$ *in* $\mathbf{R}_z$ *with each* H_j *squarefree and all divisors intersecting transversely at* z*. Suppose that* G *is holomorphic in a neighborhood of* z *and* $G(z) \neq 0$*. Then*

$$\frac{1}{(2\pi i)^d} \int_T z^{-r-1} F(z)\, dz \sim \Phi_z(\mathbf{r})$$

with

$$\Phi_z(\mathbf{r}) := \frac{1}{(\boldsymbol{m}-1)!} \frac{z^{-\mathbf{r}} G(z)}{\det \Gamma_\Psi(z)} (\mathbf{r}\Gamma_\Psi^{-1})^{\boldsymbol{m}-1} . \tag{10.3.3}$$

The remainder term is of a lower exponential order, $\exp[|\mathbf{r}|(\hat{\mathbf{r}} \cdot \log z - \varepsilon)]$*, uniformly as* $\hat{\mathbf{r}}$ *varies over compact subsets of the interior of* $\mathbf{N}(z)$.

Γ_Ψ is the matrix whose rows are the logarithmic gradients $\nabla_{\log} H_j(z)$:

$$\Gamma_\Psi = \left[x_j \frac{\partial H_i}{\partial x_j} \right]_{i,j}$$

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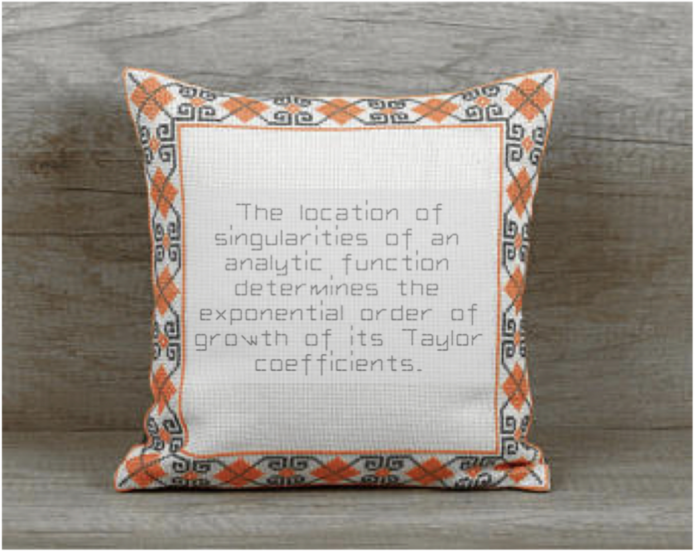
Three critical points:

$$(w, w^2, w^2/3), (w^2, w, w/3), (1, 1, 1/3)$$

$$[z^n]T(z) = 3^n \cdot n^{-3/2} \cdot \frac{3\sqrt{3}}{4\sqrt{\pi}} + O(3^n \cdot n^{-5/2})$$

The two smooth critical points give a contribution $O(3^n/n^2)$.

Two Main Takeaway Ideas


The location of
singularities of an
analytic function
determines the
exponential order of
growth of its Taylor
coefficients.


The nature of the singularities determines the way the dominant exponential term in coefficients is modulated by a subexponential factor.

Summary

Diagonal Asymptotics

Given:

$$F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum f(\mathbf{n})\mathbf{z}^{\mathrm{n}}$$

Determine the asymptotics of $f(n, n, \ldots, n)$ as $n \to \infty$

- Singular Variety $\mathcal{V} = \{\mathbf{z} \mid H(\mathbf{z}) = 0\}$
- Minimal Points: $\partial\mathcal{D} \cap \mathcal{V}$
- Critical points minimize: $|\rho_1 \ldots \rho_d|^{-1}$ (with value μ, say)
- Minimal critical point ρ contained in both

$$\underbrace{-\log|z_1| - \cdots - \log|z_d| = \log\mu}_{\text{a hyperplane}} \qquad \rho \in \partial\mathcal{D} \cap \mathcal{V}$$

- Treat each critical point: If variety is smooth at critical point, THM A; Transverse multiple point THM B; Else...

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