Counting plane lattice walks avoiding a quadrant

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Counting walks in (rational) cones

Take a starting point $p_0$ in $\mathbb{Z}^2$, a (finite) step set $S \subset \mathbb{Z}^2$ and a cone $C$.

Questions

- What is the number $c(n)$ of $n$-step walks starting at $p_0$, taking their steps in $S$ and contained in $C$?
- For $(i, j) \in C$, what is the number $c(i, j; n)$ of such walks that end at $(i, j)$?
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• Generating function:

$$C(x, y; t) = \sum_{i, j, n} c(i, j; n)x^iy^jt^n$$

$$= \sum_w x^{i(w)}y^{j(w)}t^{|w|}$$

What is the value/nature of this series?
A hierarchy of formal power series

- Rational series
  \[ A(t) = \frac{P(t)}{Q(t)} \]
- Algebraic series
  \[ P(t, A(t)) = 0 \]
- Differentially finite series (D-finite)
  \[ \sum_{i=0}^{d} P_i(t) A^{(i)}(t) = 0 \]
- D-algebraic series
  \[ P(t, A(t), A'(t), \ldots, A^{(d)}(t)) = 0 \]
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Multi-variate series: one DE per variable
Normalizing the cone: 4 cases

- The full space: rational series

\[ C(x, y; t) = \frac{1}{1 - tS(x, y)} = \sum_{n \geq 0} t^n S(x, y)^n, \]

where \( S(x, y) \) is the step polynomial:

\[ S(x, y) = \sum_{(i,j) \in S} x^i y^j. \]
Normalizing the cone: 4 cases

- The full space: rational series
- A half-space: algebraic series

  [Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...
Normalizing the cone: 4 cases

- The full space: rational series
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  [Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...
- A convex cone → walks in the non-negative quadrant: \( Q(x, y; t) \)
Normalizing the cone: 4 cases

- The full space: **rational** series

- A half-space: **algebraic** series
  [Gessel 80]; [mbm-Petkovšek 00], [Duchon 00], [Banderier & Flajolet 02]...

- A convex cone → walks **in the non-negative quadrant**: $Q(x, y; t)$

- A non-convex cone → walks **avoiding** the negative quadrant: $C(x, y; t)$
Walks with *small* steps

- $S \subset \mathbb{Z}^2 \setminus \{(0,0)\} \Rightarrow 2^8 = 256$ step sets (or: *models*)

However, some models are equivalent to a half-space problem (hence algebraic) and/or to another model (diagonal symmetry).

On the quadrant, one is left with 79 interesting distinct models [mbm-Mishna 09].

On the three-quadrant cone, one is left with 74 interesting distinct models: the 5 "singular" models on the quadrant become trivial.

Singular models
Walks with small steps

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Walks with small steps

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Singular models

\begin{align*}
\begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
  & & & \\
\end{array}
\end{align*}
Non-singular

Singular
I. Functional equations

A step by step construction of walks

\[(i, j) = (5, 1)\]
Example: \( \mathcal{S} = \{01, \bar{1}0, 1\bar{1}\} \), with \( \bar{x} := 1/x \) and \( \bar{y} := 1/y \)

\[
Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)
\]

\[
Q(x, y; t) = \sum_{i,j,n \geq 0} q(i, j; n) x^i y^j t^n
\]
In the quadrant

Example: $S = \{01, \bar{1}0, 1\bar{1}\}$, with $\bar{x} := 1/x$ and $\bar{y} := 1/y$

$$Q(x, y; t) \equiv Q(x, y) = 1 + t(y + \bar{x} + x\bar{y})Q(x, y) - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0)$$

or

$$(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),$$
In the quadrant

Example: \( S = \{01, \bar{1}0, 1\bar{1}\} \), with \( \bar{x} := 1/x \) and \( \bar{y} := 1/y \)

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\]

or

\[
(1 - t(y + \bar{x} + x\bar{y}))Q(x, y) = 1 - t\bar{x}Q(0, y) - tx\bar{y}Q(x, 0),
\]

or

\[
(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)
\]

• The right-hand side is decoupled in \( x/y \).

• The polynomial \( 1 - t(y + \bar{x} + x\bar{y}) \) is the kernel of this equation

• The equation involves two catalytic variables \( x \) and \( y \) (tautological at \( x = 0 \) or \( y = 0 \))
In three quadrants

Step by step construction:

\[ C(x, y; t) \equiv C(x, y) = 1 + t(y + \bar{x} + x\bar{y})C(x, y) - t\bar{x}C_{0,-}(\bar{y}) - tx\bar{y}C_{-,0}(\bar{x}) \]

with

\[ C_{0,-}(\bar{y}) = \sum_{j<0, n \geq 0} c(0, j; n)y^{j}t^{n}, \quad C_{-,0}(\bar{x}) = \sum_{i<0, n \geq 0} c(i, 0; n)x^{i}t^{n}. \]

\[ C(x, y; t) = \sum_{i, j, n} c(i, j; n)x^{i}y^{j}t^{n} \]
In three quadrants

Step by step construction:

\[ C(x, y; t) \equiv C(x, y) = 1 + t(y + \bar{x} + x\bar{y})C(x, y) - t\bar{x}C_{0, -}(\bar{y}) - t\bar{y}C_{-, 0}(\bar{x}) \]

with

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or

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or

\[ (1 - t(y + \bar{x} + x\bar{y})) C(x, y) = 1 - t\bar{x}C_{0, -}(\bar{y}) - tx\bar{y}C_{-0}(\bar{x}), \]

or

\[ (1 - t(y + \bar{x} + x\bar{y})) xyC(x, y) = xy - tyC_{0, -}(\bar{y}) - tx^2 C_{-0}(\bar{x}). \]
A comparison

- First quadrant:

\[
(1 - t(y + \bar{x} + x\bar{y})) \, xyQ(x, y) = xy - tyQ(0, y) - tx^2 Q(x, 0)
\]

- Three quadrants:

\[
(1 - t(y + \bar{x} + x\bar{y})) \, xyC(x, y) = xy - tyC_{0,-}(\bar{y}) - tx^2 C_{-,0}(\bar{x})
\]

with

\[
C_{0,-}(\bar{y}) = \sum_{j<0, n\geq 0} c(0, j; n) y^j t^n, \quad C_{-,0}(\bar{x}) = \sum_{i<0, n\geq 0} c(i, 0; n) x^i t^n.
\]

- A similar form... but \( C(x, y) \) involves negative powers of \( x \) and \( y \)
  (Laurent polynomials)
II. The group of the model and the orbit sum
Example. Take $S = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$
The group of the model

**Example.** Take $S = \{\bar{1}0, 01, 1\bar{1}\}$, with step polynomial

$$S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}$$

**Observation:** $S(x, y)$ is left unchanged by the rational transformations

$$\Phi : (x, y) \mapsto (\bar{x}y, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y})$$
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They are involutions, and generate a finite dihedral group \( G \):

\[
\begin{align*}
\Phi & : (x, y) \mapsto (\bar{y}x, y) \\
\Psi & : (x, y) \mapsto (x, x\bar{y}) \\
\Phi & : (\bar{y}x, y) \mapsto (y, \bar{x}) \\
\Phi & : (x, x\bar{y}) \mapsto (\bar{y}, x\bar{y}) \\
\end{align*}
\]
The group of the model

Example. Take \( S = \{\overline{10}, 01, 1\overline{1}\} \), with step polynomial

\[
S(x, y) = \frac{1}{x} + y + \frac{x}{y} = \bar{x} + y + x\bar{y}
\]

Observation: \( S(x, y) \) is left unchanged by the rational transformations

\[
\Phi : (x, y) \mapsto (\bar{y}x, y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, x\bar{y}).
\]

They are involutions, and generate a finite dihedral group \( G \):

The group \( G \) can be defined for any model with small steps.
The group is not always finite

- If \( S = \{0\overline{1}, \overline{1}, \overline{10}, 11\} \), then \( S(x, y) = \overline{x}(1 + \overline{y}) + \overline{y} + xy \) and

  \[
  \Phi : (x, y) \mapsto (\overline{xy}(1 + \overline{y}), y) \quad \text{and} \quad \Psi : (x, y) \mapsto (x, \overline{xy}(1 + \overline{x}))
  \]

generate an infinite group:
• The quadrant equation reads (with $K(x, y) = 1 - t(y + \bar{x} + x\bar{y})$):

$$K(x, y)xyQ(x, y) = xy - tx^2 Q(x, 0) - tyQ(0, y)$$

• The orbit of $(x, y)$ under $G$ is

$$(x, y) \xrightarrow{\Phi} (\bar{x}y, y) \xrightarrow{\Psi} (\bar{x}y, \bar{x}) \xrightarrow{\Phi} (\bar{y}, \bar{x}) \xrightarrow{\Psi} (\bar{y}, x\bar{y}) \xrightarrow{\Phi} (x, x\bar{y}) \xrightarrow{\Psi} (x, y).$$
The orbit sum

- The quadrant equation reads (with $K(x, y) = 1 - t(y + x + x\bar{y})$):
  
  $$K(x, y)xyQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$

- The orbit of $(x, y)$ under $G$ is
  
  $$(x, y)\xrightarrow{\Phi}(\bar{x}y, y)\xrightarrow{\Psi}(\bar{x}y, \bar{x})\xrightarrow{\Phi}(\bar{y}, \bar{x})\xrightarrow{\Psi}(\bar{y}, x\bar{y})\xrightarrow{\Phi}(x, x\bar{y})\xrightarrow{\Psi}(x, y).$$

- All transformations of $G$ leave $K(x, y)$ invariant. Hence
  
  $$K(x, y) \times yQ(x, y) = xy - tx^2Q(x, 0) - tyQ(0, y)$$
  $$K(x, y) \bar{x}y^2Q(\bar{x}y, y) = \bar{x}y^2 - t\bar{x}^2y^2Q(\bar{x}y, 0) - tyQ(0, y)$$
  $$K(x, y) \bar{x}^2yQ(\bar{x}y, \bar{x}) = \bar{x}^2y - t\bar{x}^2y^2Q(\bar{x}y, 0) - t\bar{x}Q(0, \bar{x})$$
  $$\ldots = \ldots$$
  $$K(x, y) x^2\bar{y}Q(x, x\bar{y}) = x^2\bar{y} - tx^2Q(x, 0) - tx\bar{y}Q(0, x\bar{y}).$$
The orbit sum

⇒ Form the alternating sum of the equation over all elements of the orbit:

\[
K(x, y) \left( xyQ(x, y) - \bar{x}y^2Q(\bar{x}y, y) + \bar{x}^2yQ(\bar{x}y, \bar{x}) \\
- \bar{x}\bar{y}Q(\bar{y}, \bar{x}) + xy^2Q(\bar{y}, x\bar{y}) - x^2\bar{y}Q(x, x\bar{y}) \right) =

xy - \bar{x}y^2 + \bar{x}^2y - \bar{x}\bar{y} + xy^2 - x^2\bar{y}

the orbit sum.
The orbit sum

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\[
K(x, y) \left( xyQ(x, y) - \bar{x} y^2 Q(\bar{x}y, y) + \bar{x}^2 y Q(\bar{x}y, \bar{x}) \\
- \bar{x} \bar{y} Q(\bar{y}, \bar{x}) + x \bar{y}^2 Q(\bar{y}, x\bar{y}) - x^2 \bar{y} Q(x, x\bar{y}) \right) =
\]

\[
xy - \bar{x} y^2 + \bar{x}^2 y - \bar{x} \bar{y} + x \bar{y}^2 - x^2 \bar{y}
\]

the orbit sum.

Similarly, for walks avoiding a quadrant:

\[
K(x, y) \left( xyC(x, y) - \bar{x} y^2 C(\bar{x}y, y) + \bar{x}^2 y C(\bar{x}y, \bar{x}) \\
- \bar{x} \bar{y} C(\bar{y}, \bar{x}) + x \bar{y}^2 C(\bar{y}, x\bar{y}) - x^2 \bar{y} C(x, x\bar{y}) \right) =
\]

\[
xy - \bar{x} y^2 + \bar{x}^2 y - \bar{x} \bar{y} + x \bar{y}^2 - x^2 \bar{y}
\]
III. Classification of walks in the first quadrant
Some examples

Algebraic [Kreweras 65, Gessel 86]

\[(1 - t(\bar{x} + \bar{y} + xy))xyQ(x, y) = xy - tyQ(0, y) - txQ(x, 0)\]

D-finite, but transcendental [Gessel 90]

\[(1 - t(y + \bar{x} + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)\]

D-algebraic, but not D-finite [Bernardi, mbm & Raschel 17]

\[(1 - t(x + \bar{x} + y + x\bar{y}))xyQ(x, y) = xy - tyQ(0, y) - tx^2Q(x, 0)\]

Not D-algebraic [Dreyfus, Hardouin, Roques & Singer 17]

\[(1 - t(x\bar{y} + \bar{x} + \bar{y} + y))xyQ(x, y) = xy - tyQ(0, y) - tx(1 + x)Q(x, 0)\]
Classification of quadrant walks

quadrant models: 79

$|G| < \infty$: 23
  \[
  \begin{align*}
  \text{D-finite} & \quad \text{OS} = 0: 4 \\
  & \quad \text{OS} \neq 0: 19 \\
  \text{algebraic} & \quad \text{DF transc.}
  \end{align*}
  \]

$|G| = \infty$: 56
  \[
  \begin{align*}
  \text{Not D-finite} & \quad \text{decoupled: 9 } \quad \text{not decoupled: 47} \\
  \text{D-alg.} & \quad \text{not D-alg.}
  \end{align*}
  \]
The finite group case (23 models)

• For the 19 models for which the orbit sum is non-zero:

\[ xyQ(x, y; t) = [x^0 y^0] \frac{OS}{K(x, y; t)} \]

is a D-finite series. [mbm-Mishna 10]

• 4 models have a vanishing orbit sum:

They all have an algebraic generating function \( Q(x, y; t) \)

[Gessel 86, Mishna 08, mbm-Mishna 10, Bostan & Kauers 10], and more!
A solution for some models
- relating $C(x, y)$ to the quadrant series $Q(x, y)$
- ...
- do three-quadrant equations with orbit sum zero have algebraic solutions?
A conjectural classification

Walks with small steps avoiding the negative quadrant:

- **models**: 74
  - $|G| < \infty$: 23
  - $|G| = \infty$: 51
    - OS $\neq 0$: 19
    - OS $= 0$: 4
      - Not D-finite

- D-finite? algebraic?

- Non-D-finiteness is proved by an asymptotic argument [Mustapha 19]
Some references

• **Explicit D-finite** expression for 2 step sets [mbm 16], [Budd 17(a)]

• **Explicit D-finite** expression for the king’s walk [mbm & Wallner 19+]

• **Integral expressions** for some symmetric models [Raschel-Trotignon 18(a)]
Example. The number of walks of length $2n$ on the diagonal square lattice starting and ending at $(0,0)$ and avoiding the negative quadrant is

$$c(0,0; 2n) = \frac{16^n}{9} \left( 3 \frac{(1/2)_n^2}{(2)_n^2} + 8 \frac{(1/2)_n(7/6)_n}{(2)_n(4/3)_n} - 2 \frac{(1/2)_n(5/6)_n}{(2)_n(5/3)_n} \right)$$

with $(a)_n = a(a + 1) \cdots (a + n - 1)$. 
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with $(a)_n = a(a + 1) \cdots (a + n - 1)$.

- The first term is $\frac{1}{3} q(0, 0; 2n)$.
- The generating functions of the other two terms are algebraic.
• The quadrant series $Q(x, y) \equiv Q(x, y; t)$ is the non-negative part in $x$ and $y$ of an explicit rational series (hence D-finite).
Explicit expressions for series

- The quadrant series $Q(x, y) \equiv Q(x, y; t)$ is the non-negative part in $x$ and $y$ of an explicit rational series (hence D-finite).

A structured series [mbm 15(a), mbm & Wallner 19+]

The series $C(x, y; t) \equiv C(x, y)$ counting walks that avoid the negative quadrant is

$$\frac{1}{3} \left( Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right) + A(x, y)$$

where $\bar{x} = 1/x$, $\bar{y} = 1/y$ and $A(x, y)$ is algebraic. This series satisfies

$$(1 - t(x + \bar{x} + y + \bar{y})) A(x, y) = (2 + \bar{x}^2 + \bar{y}^2)/3 - t\bar{y} A_-(\bar{x}) - t\bar{x} A_-(\bar{y}),$$

where $A_-(x)$ is a series in $t$ with coefficients in $\mathbb{Q}[x]$, algebraic of degree $24/24/72$, given explicitly with its intermediate extensions.
The solution:

\[
C(x, y) = \frac{1}{3} \left( Q(x, y) - \bar{x}^2 Q(\bar{x}, y) - \bar{y}^2 Q(x, \bar{y}) \right) + A(x, y)
\]

where \( A(x, y) \) is algebraic.

- Where the series \( A(x, y) \) comes from
- Polynomial equations with one catalytic variable have algebraic solutions
Where the series $A(x, y)$ comes from

• The group: $(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$
Where the series $A(x, y)$ comes from

- The group: $(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$

- The orbit sum:
  \[
  (1 - t(x + \bar{x} + y + \bar{y})) \left( xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) \right) \\
  = (1 - t(x + \bar{x} + y + \bar{y})) \left( xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) \right) \\
  = xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.
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  = xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.
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- The alternating sums of the following series are the same:
  \[
  xyC(x, y), \quad xyQ(x, y),
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Where the series $A(x, y)$ comes from

- The group: $(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$

- The orbit sum:
  \[
  (1 - t(x + \bar{x} + y + \bar{y})) \left( xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) \right) \\
  = (1 - t(x + \bar{x} + y + \bar{y})) \left( xyQ(x, y) - \bar{x}yQ(\bar{x}, y) + \bar{x}\bar{y}Q(\bar{x}, \bar{y}) - x\bar{y}Q(x, \bar{y}) \right) \\
  = xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}.
  \]

- The alternating sums of the following series are the same:
  $xyC(x, y), \quad xyQ(x, y), \quad$ but also $-\bar{x}yQ(\bar{x}, y), \quad \bar{x}\bar{y}Q(\bar{x}, \bar{y}), \quad -x\bar{y}Q(x, \bar{y})$
Where the series $A(x, y)$ comes from

- The group: $(x, y), (\bar{x}, y), (\bar{x}, \bar{y}), (x, \bar{y})$
- The orbit sum:

\[
(1 - t(x + \bar{x} + y + \bar{y})) \left( \left( xyC(x, y) - \bar{x}yC(\bar{x}, y) + \bar{x}\bar{y}C(\bar{x}, \bar{y}) - x\bar{y}C(x, \bar{y}) \right) 
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= (1 - t(x + \bar{x} + y + \bar{y})) \left( xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y} \right).
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- The alternating sums of the following series are the same:

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- Define $A(x, y)$ by

\[
xyC(x, y) := xyA(x, y) + \frac{1}{3} \left( xyQ(x, y) - \bar{x}yQ(\bar{x}, y) - x\bar{y}Q(x, \bar{y}) \right).
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  \]
Three-quadrant equations with vanishing orbit sum: algebraicity?

• The series $A(x, y)$ has orbit sum zero and satisfies

$$(1 - t(x + \bar{x} + y + \bar{y}))xyA(x, y) = (2xy + \bar{x}y + x\bar{y})/3 - tyA_-(\bar{y}) - txA_-(\bar{x})$$

**Thm.** The series $A(x, y)$ is algebraic for the three models.

[mbm 16, mbm & Wallner 19+]
Three-quadrant equations with vanishing orbit sum: algebraicity?

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**Thm.** The series $A(x, y)$ is algebraic for the three models.

- Another indication: for three-quadrant walks on the square lattice starting at $(-1, 0)$, with equation
  \[(1 - t(x + \bar{x} + y + \bar{y}))xyC(x, y) = y - tyC_{0,-}(\bar{y}) - txC_{-,0}(x).\]

The orbit sum is $y - y + \bar{y} - \bar{y} = 0$ and $C(x, y)$ is algebraic [mbm 16].
A general structure? Some conjectures

- The 4 models with orbit sum zero have an algebraic generating function:

\[
\text{For any "reflection" model, }
xyC(x, y) = xyA(x, y) + 1 - |G| - 1 \sum_{g \in G, g \neq g_{\text{max}}} \text{sign}(g) g(xyQ(x, y))
\]

where \(A(x, y)\) is algebraic.

- What about the following vertically symmetric models?
A general structure? Some conjectures

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DF
Back to series with *polynomial coefficients* in $x$ and $y$

- The equation for $A(x, y)$ (with $K(x, y) = 1 - t(x + \bar{x} + y + \bar{y})$):

  $$K(x, y)A(x, y) = \frac{2 + \bar{x}^2 + \bar{y}^2}{3} - t\bar{y}A_-(\bar{x}) - t\bar{x}A_-(\bar{y})$$

- Split $A(x, y)$ into three parts:

  $$A(x, y) = P(x, y) + \bar{x}M(\bar{x}, y) + \bar{y}M(\bar{y}, x)$$

$\Rightarrow$ Equations for $P(x, y)$ and $M(x, y)$
The equation for $M(x, y)$ is (almost) a quadrant equation

- $M(x, y)$ is a series in $t$ with coefficients in $\mathbb{Q}[x, y]$, satisfying

$$
(1 - t(x + \bar{x} + y + \bar{y})) \left(2M(x, y) - M(0, y)\right) = \\
2x/3 - 2t\bar{y}M(x, 0) + t(x - \bar{x})M(0, y) + t\bar{y}M(y, 0).
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\]

- Proof that $M(x, y)$ is algebraic:
  - [...]  
  - equation for $M(0, x)$: let $S(x) = txM(0, x)$, then

\[
((1 - t(x + \bar{x}))^2 - 4t^2) \left( S(x)^3 + \frac{2x + 1}{x + 1} S(x)^2 + \frac{S(x)}{\bar{x} + 1} \right) =
\]

\[
(t^2(x + \bar{x}) - A_2) \left( S(x) + 1 \right) + t^2 A_1 - \frac{2t^2 A_1 - A_2}{x + 1} - t^2 \bar{x},
\]

where $A_1$ and $A_2$ are series in $t$ only.

A polynomial equation with one catalytic variable $(x)$ only.
Polynomial equations with one catalytic variable

Theorem [mbm-Jehanne 06]

Let \( P(t, x, S(x; t), A_1(t), \ldots, A_k(t)) \) be a proper polynomial equation in one catalytic variable \( y \) (it defines uniquely \( S(x; t), A_1(t), \ldots, A_k(t) \) as formal power series in \( t/x \)). Then each of these series is algebraic.

The proof is constructive.

Example:

\[
((1 - t(x + \bar{x}))^2 - 4t^2) \left( \frac{S(x)^3}{x + 1} + \frac{2x + 1}{x + 1} S(x)^2 + \frac{S(x)}{\bar{x} + 1} \right) =
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Algebraicity follows from (a special case of) an Artin approximation theorem with “nested” conditions [Popescu 86, Swan 98]
Computer algebra to the rescue!

- The final equation with one catalytic variable

\[ P(t, x, S(x; t), A_1(t), \ldots, A_k(t)) = 0 \]

- involve \( k = 4 \) series \( A_i \) (instead of \( k = 2 \) for the square lattice)
- they are given by a big system of 4 polynomial equations, solved (partly) by guess & check
- each \( A_i \) ends up having degree 24 (instead of 8)
- their common genus is 2 instead of 0 (\( \Rightarrow \) normal hyperelliptic forms)

[mbm & Wallner 19+]
An example

• The generating function of walks ending at \((-1, 0)\) is algebraic of degree 24:

\[
\sum_n c(-1, 0; n)t^n = \frac{1}{2t} \left( \frac{1 + 2V}{2V^3 - 4V - 1} W - 1 \right)
\]

where

\[
U = t(1 + 18U^2 - 27U^4) + t^2(1 + U)(1 - 3U)^3,
\]

\[
V = U \frac{1 + 3V - V^3}{1 + V + V^2},
\]

\[
W^2 = 1 + 4V - 4V^3 - 4V^4.
\]
Further questions

- Complete the classification...
- First for models such that an (algebraic?) series $A(x, y)$ can be defined by

$$xyC(x, y) = xyA(x, y) + \frac{1}{|G| - 1} \sum_{g \in G, g \neq g_{\text{max}}} \text{sign}(g)g(xyQ(x, y))$$

- What about the other models with a finite group?
- Are there $D$-algebraic (but not $D$-finite) models?

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models: 74
| G | <\infty: 23 | G | =\infty: 51
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