Partition identities of Capparelli and Primc

Jehanne Dousse

CNRS and Université Lyon 1

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Outline

Integer partitions

Definition

A *partition* π of a positive integer n is a finite non-increasing sequence of positive integers $\lambda_1, \ldots, \lambda_m$ such that $\lambda_1 + \cdots + \lambda_m = n$. The integers $\lambda_1, \ldots, \lambda_m$ are called the *parts* of the partition.

Example

There are 5 partitions of 4:

$$4, 3+1, 2+2, 2+1+1 \text{ and } 1+1+1+1.$$

Generating functions

Notation : $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), n \in \mathbb{N} \cup \{\infty\}$.

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Let $Q(n,k)$ be the number of partitions of n into k distinct parts. Then

$$\begin{aligned} 1 + \sum_{n \geq 1} \sum_{k \geq 1} Q(n,k) z^k q^n &= (1+zq)(1+zq^2)(1+zq^3)(1+zq^4)\cdots \\ &= (-zq;q)_\infty . \end{aligned}$$

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Let $p(n,k)$ be the number of partitions of n into k parts. Then

$$\begin{aligned} 1+\sum_{n\geq 1}\sum_{k\geq 1} p(n,k)z^k q^n &= \prod_{n\geq 1}\left(1+zq^n+z^2q^{2n}+\cdots\right) \\ &= \frac{1}{(zq;q)_\infty}. \end{aligned}$$

Generating functions

More generally:

- The generating function for partitions into distinct parts congruent to $k \mod N$ is
$$(-zq^k;q^N)_\infty.$$
- The generating function for partitions into parts congruent to $k \mod N$ is
$$\frac{1}{(zq^k;q^N)_\infty}.$$

So the general shape of a generating function for partitions with congruence conditions is

$$\frac{(-z_1q^{k_1};q^{N_1})_\infty \cdots (-z_sq^{k_s};q^{N_s})_\infty}{(z'_1q^{k'_1};q^{N'_1})_\infty \cdots (z'_rq^{k'_r};q^{N'_r})_\infty}.$$

Partition identities

Theorem (Euler 1748)

For every integer n, the number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

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Proof.

$$\begin{aligned}\prod_{n\geq 1}(1+q^n) &= \prod_{n\geq 1}\frac{(1+q^n)(1-q^n)}{1-q^n}\\ &= \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^n}\\ &= \prod_{n\geq 1}\frac{1}{1-q^{2n-1}}.\end{aligned}$$

□

The first Rogers-Ramanujan identity

Theorem (Rogers 1894, Rogers-Ramanujan 1919)

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty},$$

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Theorem (Partition version)

For every positive integer n, the number of partitions of n such that the difference between two consecutive parts is at least 2 is equal to the number of partitions of n into parts congruent to 1 or 4 modulo 5.

Representation theoretic interpretation

Lepowsky and Wilson 1984: representation theoretic interpretation

$$\frac{1}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q;q^2)_\infty}\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}$$

RHS: principal specialized Weyl-Kac character formula of standard $A_1^{(1)}$-modules of level 3

LHS comes from bases of level 3 standard $A_1^{(1)}$-modules constructed from vertex operators

Some other identities from representation theory

Studying other representations or other Lie algebras lead to new identities:

- Capparelli 1993: level 3 standard modules of $A_2^{(2)}$
- Nandi 2014: level 4 standard modules of $A_2^{(2)}$
- Meurman and Primc 1987-1999: higher levels of $A_1^{(1)}$
- Siladić 2002: twisted level 1 modules of $A_2^{(2)}$
- Primc 1999: $A_2^{(1)}$ and $A_1^{(1)}$ crystals
- Primc and Šikić 2016: level k standard modules of $C_n^{(1)}$

Outline

Capparelli's identity

From the study of level 3 standard modules of $A_2^{(2)}$:

Theorem (Capparelli (conj. 1992, proof 1994), Andrews 1992)

Let $C(n)$ denote the number of partitions of n into distinct parts congruent to $0, 2, 3, 4 \mod 6$.
Let $D(n)$ denote the number of partitions $\lambda_1 + \cdots + \lambda_s$ of n such that $\lambda_s \neq 1$ and

$$\lambda_i - \lambda_{i+1} \geq \begin{cases} 2 & \text{if } \lambda_i, \lambda_{i+1} \equiv 0 \mod 3 \text{ or } \lambda_i + \lambda_{i+1} \equiv 0 \mod 6 \\ 4 & \text{otherwise.} \end{cases}$$

Then for all n, $C(n) = D(n)$.

Example

The partitions counted by $C(9)$ are 9, $6+3$, and $4+3+2$.
The partitions counted by $D(9)$ are 9, $7+2$ and $6+3$.

Non-dilated version (method of weighted words)

- Consider partitions into coloured integers

$$2_b < 1_c < 2_a < 3_b < 2_c < 3_a < 4_b < 3_c < \cdots ,$$

satisfying the difference conditions

$$\lambda_i - \lambda_{i+1} \geq D(color(\lambda_i), color(\lambda_{i+1})),$$

where D is the following matrix

$$D = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 2 & 0 & 2 \\ 2 & 2 & 3 \\ 1 & 0 & 1 \end{pmatrix} \end{array}.$$

After performing the transformations

$$k_c \mapsto 3k, k_a \mapsto 3k-2, k_b \mapsto 3k-4,$$

these partitions satisfy the difference conditions of Capparelli's identity.

Non-dilated version (method of weighted words)

- Compute "directly" generating function for $D(n; i, j, k)$, the number of partitions of n with i parts coloured a, j parts coloured b and k parts coloured c, satisfying the difference conditions from matrix D.

$$\sum_{i,j,k,n \geq 0} D(n; i, j, k) a^i b^j c^k q^n = \sum_{i,j \geq 0} \frac{a^i b^j q^{2\binom{i+1}{2} + 2\binom{j+1}{2}} (-q; q)_{i+j} (-cq^{i+j+1}, q)_\infty}{(q^2; q^2)_i (q^2; q^2)_j}.$$

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- Using q-series identities, we show that this is a suitable infinite product if and only if $c = 1$, and in that case it equals

$$(-q;q)_\infty(-aq^2;q^2)_\infty(-bq^2;q^2)_\infty.$$

Non-dilated version (Alladi-Andrews-Gordon 1993)

Capparelli's identity, non-dilated version

Let $D(n;i,j)$ denote the number of coloured partitions of n with i parts coloured a and j parts coloured b such that there is no part 1_a or 1_b, satisfying the difference conditions from matrix D. Then we have

$$\sum D(n;i,j)a^i b^j q^n = (-q;q)_\infty(-aq^2;q^2)_\infty(-bq^2;q^2)_\infty.$$

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The dilation $q \to q^3, a \to aq^{-2}, b \to bq^{-4}$ gives a refinement of Capparelli's identity.

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By using other dilations or changing the order on the integers, one can obtain infinitely many new partition identities.

Outline

Primc (1999): partition identity arising from crystal bases of $A_1^{(1)}$.
Partitions in four colours a, b, c, d, with the order

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots ,$$

and difference conditions

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

Conjecture (Primc 1999)

Under the dilations

$$k_a \to 2k-1, k_b \to 2k, k_c \to 2k, k_d \to 2k+1,$$

the generating function for these partitions (not keeping track of the colours) is equal to $\frac{1}{(q;q)_\infty}$.

Theorem (D.-Lovejoy 2017)

Let $A(n; k, \ell, m)$ denote the number of partitions satisfying the difference conditions of matrix P, with k parts coloured a, ℓ parts coloured c and m parts coloured d. Then

$$\sum_{n,k,\ell,m \geq 0} A(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q)_\infty (cq; q^2)_\infty}.$$

Under the dilations

$$q \to q^2, a \to aq^{-1}, b \to 1, c \to c, d \to dq,$$

the ordering of integers becomes

$$1_a < 2 < 2_c < 3_d < 3_a < 4 < 4_c < 5_d < \cdots ,$$

Theorem (Refinement of Primc's theorem)

Let $A_2(n; k, \ell, m)$ denote the number of coloured partitions of n satisfying the (dilated) difference conditions, such that odd parts can be coloured a or d and even parts can be coloured c or uncoloured, with no part 1_d, having k parts coloured a , ℓ parts coloured c and m parts coloured d. Then

$$\sum_{n,k,\ell,m \geq 0} A_2(n; k, \ell, m) q^n a^k c^\ell d^m = \frac{(-aq; q^4)_\infty (-dq^3; q^4)_\infty}{(q^2; q^2)_\infty (cq^2; q^4)_\infty}.$$

Proof: variant of the method of weighted words

$$P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

- Define $G_k^P(q;a,b,c,d)$ (resp. $E_k^P(q;a,b,c,d)$) to be the generating function for coloured partitions satisfying the difference conditions from matrix P with the added condition that the largest part is at most (resp. equal to) k. Find 4 recurrences such as

$$\begin{aligned} G_{k_d}^P(q;a,b,c,d) - G_{k_c}^P(q;a,b,c,d) &= E_{k_d}^P(q;a,b,c,d) \\ &= dq^k(E_{k_c}^P(q;a,b,c,d) + E_{k_a}^P(q;a,b,c,d) + G_{(k-1)_c}^P(q;a,b,c,d)). \end{aligned}$$

- Try to find $\lim_{k\to\infty} G_k^P(q; a, 1, c, d)$.

- Try to find $\lim_{k\to\infty} G_k^P(q; a, 1, c, d)$.

  Combine the four equations to obtain

$$(1 - cq^k) G_{k_d}^P = \frac{1 - cq^{2k}}{1 - q^k} G_{(k-1)_d}^P + \frac{aq^k + dq^k + adq^{2k}}{1 - q^{k-1}} G_{(k-2)_d}^P + \frac{adq^{2k-1}}{1 - q^{k-2}} G_{(k-3)_d}^P.$$

- Try to find $\lim_{k\to\infty} G_k^P(q;a,1,c,d)$.

  Combine the four equations to obtain

$$(1-cq^k)G^P_{k_d} = \frac{1-cq^{2k}}{1-q^k}G^P_{(k-1)_d} + \frac{aq^k+dq^k+adq^{2k}}{1-q^{k-1}}G^P_{(k-2)_d} + \frac{adq^{2k-1}}{1-q^{k-2}}G^P_{(k-3)_d}.$$

  Let

$$H_k(q;a,b,c,d) := \frac{G^P_{k_d}(q;a,b,c,d)}{1-q^{k+1}}.$$

  Then

$$(1-cq^k-q^{k+1}+cq^{2k+1})H_k = (1-cq^{2k})H_{k-1} + (aq^k+dq^k+adq^{2k})H_{k-2} + adq^{2k-1}H_{k-3}. \quad (1)$$

Now define

$$f(x) := \sum_{k\geq 0} H_{k-1}x^k,$$

and convert it into a q-difference equation on f:

$$(1-x)f(x) = (1+\frac{c}{q}+ax^2q+dx^2q)f(xq)-(1+xq)(\frac{c}{q}-adx^2q^2)f(xq^2).$$

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This is a q-difference of order 2, better but still not so easy to solve.
Define

$$g(x) := \frac{f(x)}{\prod_{k\geq 0} 1+xq^k}.$$

We obtain:

$$(1-x^2)g(x) = (1+\frac{c}{q}+ax^2q+dx^2q)g(xq)-(\frac{c}{q}-adx^2q^2)g(xq^2).$$

Let us go back to recurrence equations again: define (a_n) as

$$\sum_{n\geq 0} a_n x^n := g(x).$$

Then (a_n) satisfies

$$a_n = \frac{\left(1+aq^{n-1}\right)\left(1+dq^{n-1}\right)}{\left(1-q^n\right)\left(1-cq^{n-1}\right)} a_{n-2},$$

and the initial conditions

$$a_0 = 1, a_1 = 0.$$

Thus for all $n \geq 0$, we have

$$a_{2n} = \frac{(-aq;q^2)_n(-dq;q^2)_n}{(q^2;q^2)_n(cq;q^2)_n},$$

$$a_{2n+1} = 0.$$

We can track back these steps to find an exact expression for $H_k(q; a, b, c, d)$, and show:

$$\lim_{k\to\infty} H_k(q; a, 1, c, d) = \frac{(-aq; q^2)_\infty(-dq; q^2)_\infty}{(q; q)_\infty(cq; q^2)_\infty}.$$

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Thus:

$$\lim_{k\to\infty} G_k^P(q; a, 1, c, d) = \frac{(-aq; q^2)_\infty(-dq; q^2)_\infty}{(q; q)_\infty(cq; q^2)_\infty}.$$

Primc's identity is proved.

Exact expression for $G_k(q; a, b, c, d)$

Theorem (Finite version of Primc's identity (D. 2018))

We have, for every positive integer k,

$$G_k^P(q; a, b, c, d) = \left(1 - bq^{k+1}\right) \sum_{j=0}^{k+1} \frac{u_j(a, b, c, d) q^{\binom{k+1-j}{2}}}{(q; q)_{k+1-j}},$$

where for all $n \geq 0$,

$$u_{2n}(a, b, c, d) = (1 - b) \sum_{\ell=0}^{n} \frac{(-aq^{2\ell+1}; q^2)_{n-\ell}(-dq^{2\ell+1}; q^2)_{n-\ell}}{(bq^{2\ell}; q^2)_{n-\ell+1}(cq^{2\ell+1}; q^2)_{n-\ell}} \frac{q^{2\ell}}{(q; q)_{2\ell}},$$

and

$$u_{2n+1}(a, b, c, d) = (b-1) \sum_{\ell=0}^{n} \frac{(-aq^{2\ell+2}; q^2)_{n-\ell}(-dq^{2\ell+2}; q^2)_{n-\ell}}{(bq^{2\ell+1}; q^2)_{n-\ell+1}(cq^{2\ell+2}; q^2)_{n-\ell}} \frac{q^{2\ell+1}}{(q; q)_{2\ell+1}}.$$

Outline

Comparison

Capparelli	Primc
level 3 standard modules of $A_2^{(2)}$	crystal bases of $A_1^{(1)}$
$$\begin{array}{c|ccc} & a' & b' & c' \\ \hline a' & 2 & 0 & 2 \\ b' & 2 & 2 & 3 \\ c' & 1 & 0 & 1 \end{array}$$	$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 2 & 1 & 2 & 2 \\ b & 1 & 0 & 1 & 1 \\ c & 0 & 1 & 0 & 2 \\ d & 0 & 1 & 0 & 2 \end{array}$$
$$\frac{(-a'q^2;q^2)_\infty(-b'q^2;q^2)_\infty}{(q;q^2)_\infty}$$	$$\frac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q)_\infty(cq;q^2)_\infty}$$

Reformulation of Capparelli's identity

Consider partitions in coloured integers

$$1_a < 1_c < 1_d < 2_a < 2_c < 2_d < \cdots ,$$

satisfying the difference conditions of the matrix

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

To recover Capparelli's original identity, one should now perform the transformations

$$k_a \mapsto 3k - 1, k_c \mapsto 3k, k_d \mapsto 3k + 1.$$

Reformulation of Capparelli's identity

Consider partitions in coloured integers

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satisfying the difference conditions of the matrix

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

Theorem (Non-dilated version of Capparelli reformulated)

Let $C(n; k, m)$ denote the number of partitions satisfying the difference conditions of matrix C, with k parts coloured a and m parts coloured d.

$$\sum_{n,k,m \geq 0} C(n; k, \ell, m) q^n a^k d^m = \frac{(-aq; q^2)_\infty (-dq; q^2)_\infty}{(q; q^2)_\infty}.$$

Updated comparison

Capparelli	Primc
level 3 standard modules of $A_2^{(2)}$	crystal bases of $A_1^{(1)}$
$\begin{array}{c} \\ a \\ c \\ d \end{array}\begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$	$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}$
$\frac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q^2)_\infty}$	$\frac{(-aq;q^2)_\infty(-dq;q^2)_\infty}{(q;q)_\infty(cq;q^2)_\infty}$

Exact relation

Recall that $G_k^P(q;a,b,c,d)$ is the generating function for coloured partitions satisfying the difference conditions from matrix P with the added condition that the largest part is at most k.

Similarly, define $G_k^C(q;a,c,d)$ is the generating function for coloured partitions satisfying the difference conditions from matrix C with the added condition that the largest part is at most k.

Theorem (D. 2018)

For all positive integers k, we have

$$\frac{G_k^C(q;a,c,d)}{(cq;q)_k} = G_k^P(q;a,c,c,d).$$

Proof

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

Using the matrix C and combinatorial reasoning on the largest part,

$$\begin{aligned} G^C_{k_d} - G^C_{k_c} = E^C_{k_d} &= dq^k \left(E^C_{k_a} + G^C_{(k-1)_c} \right), \\ G^C_{k_c} - G^C_{k_a} = E^C_{k_c} &= cq^k G^C_{(k-1)_c}, \\ G^C_{k_a} - G^C_{(k-1)_d} = E^C_{k_a} &= aq^k G^C_{(k-2)_d}. \end{aligned}$$

Proof

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array}$$

Using the matrix C and combinatorial reasoning on the largest part,

$$G^C_{k_d} - G^C_{k_c} = E^C_{k_d} = dq^k \left(E^C_{k_a} + G^C_{(k-1)_c} \right),$$

$$G^C_{k_c} - G^C_{k_a} = E^C_{k_c} = cq^k G^C_{(k-1)_c},$$

$$G^C_{k_a} - G^C_{(k-1)_d} = E^C_{k_a} = aq^k G^C_{(k-2)_d}.$$

Combine these recurrences to

$$G^C_{k_d} = \left(1 + cq^k\right) G^C_{(k-1)_d} + \left(aq^k + dq^k + adq^{2k}\right) G^C_{(k-2)_d} + adq^{2k-1} \left(1 - cq^{k-1}\right) G^C_{(k-3)_d}.$$

Proof

$$G^C_{k_d} = \left(1 + cq^k\right) G^C_{(k-1)_d} + \left(aq^k + dq^k + adq^{2k}\right) G^C_{(k-2)_d} + adq^{2k-1}\left(1 - cq^{k-1}\right) G^C_{(k-3)_d}. \tag{2}$$

Recall, from the proof of Primc's identity, the function

$$H_k(q; a, b, c, d) := \frac{G^P_{k_d}(q; a, b, c, d)}{1 - q^{k+1}},$$

satisfying

$$(1 - cq^k - q^{k+1} + cq^{2k+1})H_k = (1 - cq^{2k})H_{k-1} + (aq^k + dq^k + adq^{2k})H_{k-2} + adq^{2k-1}H_{k-3}.$$

Using (2) and the initial conditions, we can show that

$$\frac{G^C_{k_d}(q; a, c, d)}{(cq; q)_{k+1}} = H_k(q; a, c, c, d). \quad \square$$

Outline

$\mathcal{C}$: set of coloured partitions satisfying the diff cond. of matrix C
$\mathcal{P}$: set of coloured partitions satisfying the diff, cond. of matrix P

Theorem (D. 2018, combinatorial version)

Let $\mathcal{C}(n;k;i,j,\ell)$ denote the set of partition pairs (λ,μ) of total weight n, where $\lambda \in \mathcal{C}$ and μ is an unrestricted partition coloured c, having in total i parts coloured a, j parts coloured c, ℓ parts coloured d, and largest part at most k.
Let $\mathcal{P}(n;k;i,j,\ell)$ denote the set of partitions $\lambda \in \mathcal{P}$ of weight n, having i parts coloured a, j parts coloured b or c, ℓ parts coloured d, and largest part at most k. Then for all positive integers n and k and all non-negative integers i,j,ℓ,

$$\#\mathcal{C}(n;k;i,j,\ell) = \#\mathcal{P}(n;k;i,j,\ell).$$

We now prove this result bijectively.

Let $(\lambda, \mu) \in \mathcal{C}(n; k; i, j, \ell)$. The partition λ satisfies the difference conditions

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array},$$

and μ is a partition coloured c.

Example

$$\lambda = 8_d + 8_a + 6_c + 5_c + 3_d + 1_a,$$
$$\mu = 8_c + 8_c + 7_c + 5_c + 3_c + 2_c + 2_c + 1_c + 1_c.$$

Step 1: Change the colour of the parts of μ to b and insert them in the partition λ according to the order of Primc's identity:

$$1_a < 1_b < 1_c < 1_d < 2_a < 2_b < 2_c < 2_d < \cdots$$

$$C = \begin{array}{c} \\ a \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccc} a & c & d \end{array} \\ \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \end{array} \longrightarrow M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \end{array}.$$

The resulting partition ν_1 satisfies the difference conditions of M_1,together with forbidden patterns

$$(m_a, m-1_a), \quad (m_c, m_a), \quad (m_c, m-1_d), \quad (m_d, m-1_d).$$

Example

$$\nu_1 = 8_d + 8_b + 8_b + 8_a + 7_b + 6_c + 5_c + 5_b + 3_d + 3_b + 2_b + 2_b + 1_b + 1_b + 1_a.$$

Step 2**:** By the difference conditions satisfied by ν_1, if m_a or m_d appears in ν_1, then m_c cannot appear, but m_b can appear arbitrarily many times. If there are such m_b's, transform them all into m_c's.

$$M_1 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \end{array} \longrightarrow M_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array},$$

The resulting partition ν_2 satisfies the difference conditions of M_2 together with forbidden patterns

$$(m_d, m_b), \quad (m_c, m-1_d),$$

and m_c can repeat if and only if it appears at the same time as m_d or m_a.

Example

$\nu_2 = 8_d + 8_c + 8_c + 8_a + 7_b + 6_c + 5_c + 5_b + 3_d + 3_c + 2_b + 2_b + 1_c + 1_c + 1_a.$

Step 3**:** If in ν_2 there is a part m_c followed by an arbitrary number of parts m_b, then change all these parts to m_c

$$M_2 = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array} \longrightarrow P = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix} \end{array}.$$

The resulting partition ν_3 satisfies the difference conditions of P.

Example

$$\nu_2 = 8_d + 8_c + 8_c + 8_a + 7_b + 6_c + 5_c + 5_c + 3_d + 3_c + 2_b + 2_b + 1_c + 1_c + 1_a.$$

All the steps are reversible.

Thank you for your attention!